Born approximation

Scattering solution using Greens function

As always we need to solve the Schrodinger equation:
$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + U(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})$$

we can rewrite it as

$$\underbrace{\nabla^2\psi(\vec{r}) + k^2\psi(\vec{r})}_{D_{\vec{r}}[\psi]} = \underbrace{\frac{2m}{\hbar^2}U(\vec{r})\psi(\vec{r})}_{f(\vec{r})}$$

We can use the Greens function method

$$\nabla^2 G(\vec{r},\vec{r}_0) + k^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$
(skipping solution...)
$$G(\vec{r}-\vec{r}_0) = -e^{ik|\vec{r}-\vec{r}_0|}/4\pi|\vec{r}-\vec{r}_0|$$

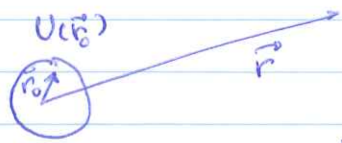
Thus, the solution of the Schrodinger equation is

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}U(\vec{r}_0)\psi(\vec{r}_0)\,d^3\vec{r}_0$$

incoming wave, solution of $-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi(\vec{r})$

This is an <u>exact</u> solution — but also not really a solution, and more like an integral equation.

Far - field region $r \gg r_0$



$$|\vec{r}-\vec{r}_0|^2 = (\vec{r}-\vec{r}_0)(\vec{r}-\vec{r}_0) = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r}_0$$
$$\approx r^2(1 - 2\vec{r}\cdot\vec{r}_0/r^2)$$

$$|\vec{r}-\vec{r}_0| = \sqrt{r^2\left(1-\frac{2\vec{r}\cdot\vec{r}_0}{r^2}\right)} \approx r\left(1-\frac{\vec{r}\,\vec{r}_0}{r^2}\right)$$

$$k\cdot|\vec{r}-\vec{r}_0| = k\cdot r - \underbrace{\left(k\,\frac{\vec{r}}{r}\right)}_{\vec{k}}\cdot\vec{r}_0 = k\cdot r - \vec{k}\cdot\vec{r}_0$$

$$\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \approx \frac{e^{ikr}}{r}\cdot e^{-i\vec{k}\cdot\vec{r}_0}$$

Thus, in the far field

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\vec{r}_0}\,U(\vec{r}_0)\psi(\vec{r}_0)\,d^3\vec{r}_0$$

~~For a plane~~ If $\psi_0(\vec{r}) = Ae^{ikz} = Ae^{i\vec{k}'\vec{r}}$ $(\vec{k}' = k\vec{e}_z)$

$$f(\theta) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\vec{r}_0}\,U(\vec{r}_0)\psi(\vec{r}_0)\,d^3\vec{r}_0$$

This is an exact (not) solution.

Born approximation: weak potential, the wave function does not change much, $\approx \psi_0(\vec{r})$ this approach is similar to the perturbation theory.

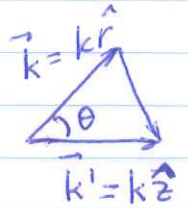
$$\psi_0(\vec{r}_0) = Ae^{i\vec{k}'\cdot\vec{r}}$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\vec{r}_0}\,U(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3\vec{r}_0$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\vec{r}_0}\,U(\vec{r}_0)\,d^3\vec{r}_0$$

That is now just an integral!

$$|\vec{q}| = |\vec{k}|^2 + |\vec{k}'|^2 - 2|\vec{k}||\vec{k}'|\cos\theta = 2k^2(1-\cos\theta)$$



$\vec{q} = \vec{k}' - \vec{k}$ momentum transfer

Initial momentum $\hbar\vec{k}'$

Final momentum $\hbar\vec{k}$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\vec{r}_0}\, U(\vec{r}_0)\, d^3\vec{r}_0$$ (we can drop "0" index $\vec{r}_0 \to \vec{r}$)

Note: $f(\theta)$ is proportional to the 3D fourier transform of the potential. Thus, to probe small-scale structure within $U(\vec{r})$, one needs ~~the~~ to have a scattering experience with high momentum transfer $(ka \gtrsim 1)$

Spherically - symmetric potential $U(\vec{r}) = U(r)$

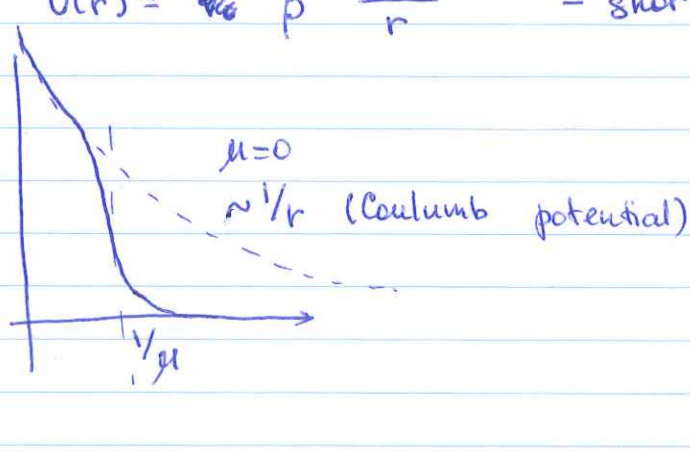
$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{iqr\cos\theta'}\, U(r)\, r^2\sin\theta'\, dr\, d\theta'\, d\varphi =$$

$$= -\frac{m}{2\pi\hbar^2}\int r^2 U(r)\, dr \int e^{iqr\cos\theta'}\, d(\cos\theta') \cdot 2\pi =$$

$$= -\frac{m}{\hbar^2}\int r^2 U(r)\, dr\, \frac{e^{iqr} - e^{-iqr}}{iqr} = -\frac{2m}{\hbar^2 q}\int_0^\infty r U(r) \sin qr\, dr$$

Example: Yukawa potential (potential describing nucleon interaction via strong force)

$$U(r) = \beta\,\frac{e^{-\mu r}}{r}$$ - short-range potential



$$f(\theta) = -\frac{2m}{\hbar^2 q} \cdot \beta \int_0^\infty e^{-\mu r} \sin qr \, dr = -\frac{2m}{\hbar^2 q} \cdot \beta \cdot \frac{1}{2i} \int_0^\infty e^{-\mu r} (e^{iqr} - e^{-iqr}) dr$$

$$= -\frac{m}{\hbar^2 q i} \beta \left[\int_0^\infty e^{-\mu r + iqr} dr - \int_0^\infty e^{-\mu r - iqr} dr \right] =$$

$$= -\frac{m}{i\hbar^2 q} \cdot \beta \left[\frac{e^{-\mu r + iqr} \Big|_0^\infty}{-\mu + iq} - \frac{e^{-\mu r - iqr} \Big|_0^\infty}{-\mu - iq} \right] =$$

$$= -\frac{m}{i\hbar^2 q} \beta \left[\frac{1}{\mu - iq} - \frac{1}{\mu + iq} \right] = -\frac{m}{\cancel{i}\hbar^2 \cancel{q}} \cdot \beta \frac{2\cancel{i}\cancel{q}}{\mu^2 + q^2}$$

$$f(\theta) = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2 + q^2} = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2 + 2k^2(1-\cos\theta)}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2\beta^2}{\hbar^4} \frac{1}{(\mu^2 + 2k^2(1-\cos\theta))^2}$$

$$\sigma = \frac{8\pi m^2 \beta^2}{\hbar^4} \int \frac{\sin\theta \, d\theta}{(\mu^2 + 2k^2(1-\cos\theta))^2} = \frac{16\pi m^2 \beta^2}{\hbar^4} \frac{1}{\mu^2(\mu^2 + 4k^2)}$$

If we are trying to calculate e scattering by a Coulumb potential, we have a problem

$$f_{Coulumb}(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty r \, U_{coulumb}(r) \sin qr \, dr = -\frac{2m}{\hbar^2 q} \int r \left(\frac{e^2}{4\pi\varepsilon_0 r} \right) \sin qr \, dr$$

$$= -\frac{2m}{\hbar^2 q} \underbrace{\int_0^\infty \sin qr \, dr}_{\text{diverge ?!}} \cdot \frac{e^2}{4\pi\varepsilon_0}$$

Instead, we'll use Yukawa potential with $\mu = 0$

$$f(\theta)_{Coulumb} = -\frac{2m}{\hbar^2} \frac{1}{2k^2(1-\cos\theta)} \left(\frac{e^2}{4\pi\varepsilon_0} \right)$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{\hbar^4 k^4} \frac{1}{(1-\cos\theta)^2} \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 = \frac{m^2}{4\hbar^4 k^4} \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{1}{\sin^4 \theta/2}$$

same as classical solution, since $E = \frac{\hbar^2 k^2}{2m}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{16}\left(\frac{2m}{\hbar^2 k^2}\right)^2 \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{\sin^4\theta/2} = \frac{1}{16E^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{\sin^4\theta/2}$$

Coinside with the classical charge on point charge scattering, and the exact solution of the quantum problem!

Can we use scattering to get information on the structure of the target

Coulumb potential → ~~[illegible]~~ $U = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$ point charge

Charge distribution → $U(\vec{r}) = \frac{e^2}{4\pi\varepsilon_0}\int \frac{\rho_N(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3\vec{r}\,'$

When substituted into the expression for $f(\theta)$

$$f(\theta) = \underbrace{-\frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{q^2}}_{f_{\text{Coulumb}}(\theta)} \underbrace{\int \rho_N(\vec{r}) e^{i\vec{q}\vec{r}} d^3\vec{r}}_{F(q)}$$

form factor

depends on the charge distribution of the (nuclea) target

For example → uniformly charged ball of radius a with density ρ_0 => $e = \frac{4\pi}{3}a^3\rho_0$

$$F(\vec{q}) = \int_0^a r^2 dr\, \rho_0 \cdot 2\pi \int_0^\pi \sin\theta\, d\theta\, e^{iqr\cos\theta} = 3\,\frac{\sin qa - aq\cos aq}{(aq)^3}$$

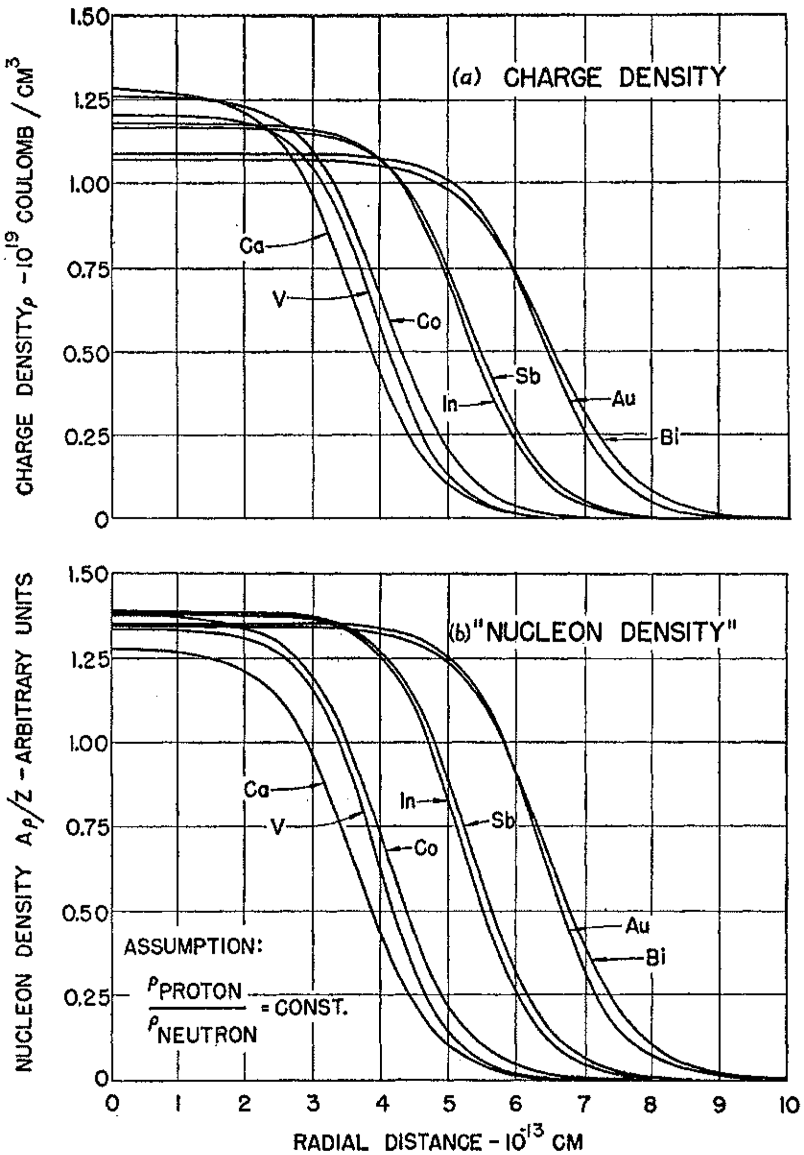


FIG. 14. (a) Charge distributions $\rho(r)$ for Ca, V, Co, In, Sb, Au, and Bi. They are Fermi smoothed uniform shapes, with the parameters given in Table III, and yield the cross sections shown in Figs. 3 and 8–12. (b) A plot of $(A/2Z)\rho(r)$ for the above nuclei. On the assumption that the distribution of matter in the nucleus is the same as the distribution of charge, this represents the "nucleon density."

Figure 2: Taken from B. Hahn, D. G. Ravenhall, and R. Hofstadter, *Phys. Rev.* **101**, 1131-1142 (1956).

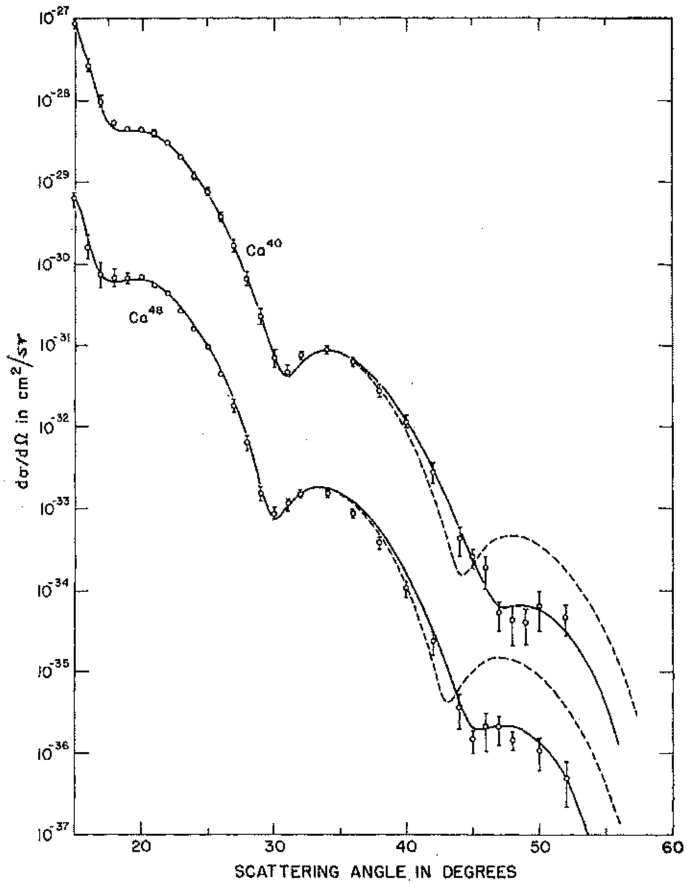
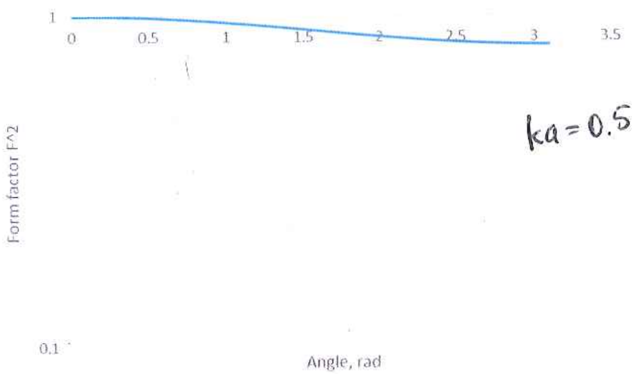


FIG. 1. Experimental and theoretical differential cross sections at 757.5 MeV. The nominal energy was 750 MeV, and a 1% adjustment was made to improve the fit at low q. The dashed curves are the best fits to earlier 250-MeV data. The charge distributions which yield them are parabolic Fermi (three-parameter) shapes [see Eq. (3) of Ref. 1] with the following parameter values: Ca^{40}, $c = 3.6685$ F, $z = 0.5839$ F, $w = -0.1017$; Ca^{48}, $c = 3.7369$ F, $z = 0.5245$ F, $w = -0.0300$. The solid curves, obtained by the method described in this Letter, come from charge distributions with an added $\Delta\rho(r)$, and parameter values $p = 0.5\ F^{-1}$, $q_0 = 3.0\ F^{-1}$, and $A(Ca^{40}) = 0.5 \times 10^{-3}$, $A(Ca^{48}) = 0.8 \times 10^{-3}$. The cross section for Ca^{40} has been multiplied by 10 and that for Ca^{48} by 10^{-1}.

Figure 1: Elastic electron scattering off calcium. Taken from J. B. Bellicard et al, *Phys. Rev. Lett.*, 19, 527 (1967)



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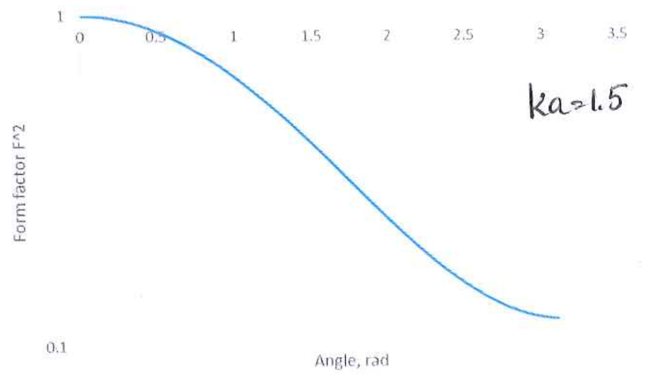
0
0.5
1
1.5
2
2.5
3
3.5



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Form factor F^2
Angle, rad
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0
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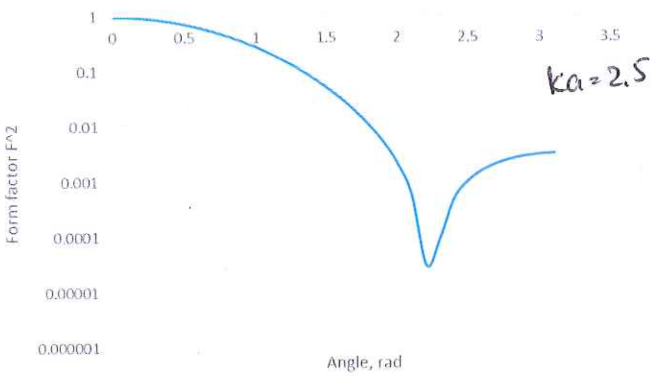
1
1.5
2
2.5
3
3.5



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0.000001
0
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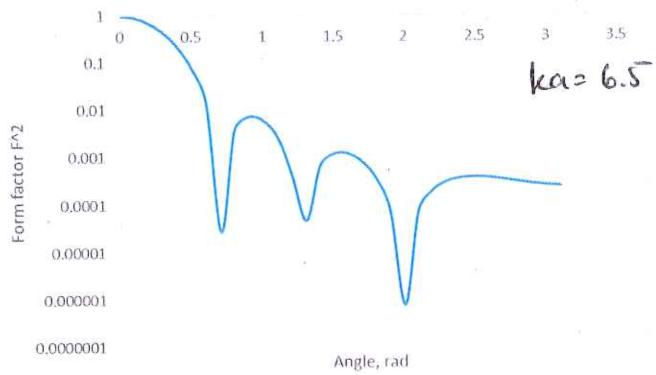
2
2.5
3
3.5



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0.001
0.0001
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0.000001
0.0000001
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1
1.5
2
2.5
3
3.5