Physics 313 Midterm test #1
October 11, 2023

Name (please print): Solutions

This test is administered under the rules and regulations of the honor system of the College of William & Mary.

Signature: ______________________________

Final score: ___________

Show all work to receive credit, and circle your final answers. This exam is closed book, but you can use a prepared index card with reference information that you have prepared.

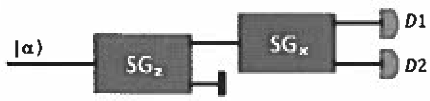
Problem 1(40 points)

A spin 1/2 particle (for example, an electron) is prepared in the quantum state $|\alpha\rangle = \sin\alpha|+z\rangle - i\cos\alpha|-z\rangle$.

(a) If a particle in this state is sent through a Stern-Gerlach apparatus that separates particles according to their S_z spin component, what are the probabilities that a particle will be measured by detectors $D1$ and $D2$?

(b) Now we have rotated the SG apparatus by 90° such that it measures S_x spin component, as shown. What are the probabilities that a particle will be measured by detectors $D1$ and $D2$?

(c) Finally, we stack two SG apparati as shown. What are the probabilities that a particle will be measured by detectors $D1$ and $D2$?



(d) Find the expectation values of spin component operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ in the state $|\alpha\rangle$.

a) D_1 probability $P_1 = |\langle +z|\alpha\rangle|^2 = \sin^2\alpha$

D_2 probability $P_2 = |\langle -z|\alpha\rangle|^2 = |-i\cos\alpha|^2 = \cos^2\alpha$

b) $P_1 = |\langle +x|\alpha\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \ \ 1)\begin{pmatrix}\sin\alpha \\ -i\cos\alpha\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}(\sin\alpha - i\cos\alpha)\right|^2 = \frac{1}{2}[\sin^2\alpha + \cos^2\alpha] = \frac{1}{2}$

$P_2 = 1 - P_1 = 1/2$

c) Probability to pass through $SG_z \rightarrow \sin^2\alpha \rightarrow$ now in $|+z\rangle$ state

——— " ——— $SG_x \rightarrow \frac{1}{2}$

Total probability $- \frac{1}{2}\sin^2\alpha$

d) $\langle S_z\rangle = \langle\alpha|\hat{S}_z|\alpha\rangle = (\sin\alpha \ \ i\cos\alpha)\frac{\hbar}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\sin\alpha \\ -i\cos\alpha\end{pmatrix} = \frac{\hbar}{2}(\sin^2\alpha - \cos^2\alpha) = -\frac{\hbar}{2}\cos 2\alpha$

$\langle S_x\rangle = \langle\alpha|\hat{S}_x|\alpha\rangle = (\sin\alpha \ \ i\cos\alpha)\frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin\alpha \\ -i\cos\alpha\end{pmatrix} = 0$

$\langle S_y\rangle = \langle\alpha|\hat{S}_y|\alpha\rangle = (\sin\alpha \ \ i\cos\alpha)\ \frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\sin\alpha \\ -i\cos\alpha\end{pmatrix} = \frac{\hbar}{2}(-2\cos\alpha\sin\alpha)$

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$= \frac{\hbar}{2}\cdot\sin 2\alpha$

Problem 2 (30 points)

The operator $\hat{F}$, acting on a spin-1 particle, in the z-basis is described by the following matrix: $\hat{F} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

Someone suggested that the following three states are the eigenstates of this operator:

$$|f_1\rangle = \mathcal{N}_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \; |f_2\rangle = \mathcal{N}_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; |f_3\rangle = \mathcal{N}_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $\mathcal{N}_i$ are the normalization constants.

(a) Determine values of $\mathcal{N}_i$ so that $|f_i\rangle$ are appropriately normalized.

(b) Check which of these states are the proper eigenstates (there may be one, two or all three). For all proper eigenstates find their eigenvalues.

(c) In part (a) you should have discovered that $|f_1\rangle$ is the eigenstate of $\hat{F}$. What is the probability that S_z is measured for a particle in the state $|f_1\rangle$, it will yield the value $-\hbar$?

(d) Circle all the vectors that also describe state $|f_1\rangle$ in the z-basis:

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \cancel{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \quad \cancel{\begin{pmatrix} i \\ 2 \\ i \end{pmatrix}} \quad \begin{pmatrix} i \\ -2i \\ i \end{pmatrix}$$

a) $\mathcal{N}_1 = \dfrac{1}{\sqrt{1^2+(-2)^2+1^2}} = \dfrac{1}{\sqrt{6}}$ $\quad \mathcal{N}_2 = \dfrac{1}{\sqrt{2}}$ $\quad \mathcal{N}_3 = \dfrac{1}{\sqrt{3}}$

b) $\hat{F}|f_1\rangle = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ eigenstate $\lambda_1 = 0$

$\hat{F}|f_2\rangle = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ not an eigenstate

$\hat{F}|f_3\rangle = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ eigenstate $\lambda_3 = 3$

c) $P = |\langle 1,-1|f_1\rangle|^2 = \left|(0\ 0\ 1)\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{6}$

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Problem 3 (30 points)

(a) Two operators, acting on spin-1/2 particle states are described by the following matrices in z-basis:

$\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $\hat{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Calculate their commutator $\hat{C} = [\hat{A}, \hat{B}]$.

(b) Calculate the uncertainties ΔA and ΔB for the $|+x\rangle$ eigenstate of $\hat{S}_x$. Check to see if the uncertainty relation $\Delta A \Delta B \geq \frac{1}{2}|\langle \hat{C} \rangle|$ is valid.

Reminder: the uncertainty of an operator is defined as $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$

a) $\hat{C} = \hat{A}\hat{B} - \hat{B}\hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

b) $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$

$\langle \hat{A} \rangle = \cancel{\frac{1}{\sqrt{2}}} \langle +x | \hat{A} | +x \rangle = \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = 2$

$\langle \hat{A}^2 \rangle = \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = 5$

$\Delta A = \sqrt{5 - 4} = 1$

$\Delta B = \sqrt{\langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2}$

$\langle \hat{B} \rangle = \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = 0$

$\langle \hat{B}^2 \rangle = \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = -1$

$\Delta B = \sqrt{-1} = i$

$\langle \hat{C} \rangle = \langle +x | \hat{C} | +x \rangle = \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = -2$

$\hat{B}$ is not a hermitian operator, so the uncertainty relation is not valid. But still $|\Delta A| \cdot |\Delta B| \geq \frac{1}{2}|\langle C \rangle|$

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$1 \cdot 1 \geq \frac{1}{2} \cdot 2$

my mistake!

Potentially useful information

Spin-1/2 particle

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Eigenstates for the spin operators:

$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix};\ |-z\rangle = \begin{pmatrix}0\\1\end{pmatrix};\ |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix};\ |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix};\ |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix};\ |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Spin-1 particle

$$\hat{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\ \hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\ \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}.$$

Eigenstates of the $\hat{S}_z$ operator (in the z-basis):

$$|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix};\ |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix};\ |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The commutator of two operators is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Potentially useful mathematical expressions

$$\begin{aligned}
&i\cdot i = -1;\ i\cdot(-i) = 1;\ 1/i = -i;\\
&e^{i\phi} = \cos\phi + i\sin\phi;\ \cos\phi = (e^{i\phi} + e^{-i\phi})/2;\ \sin\phi = (e^{i\phi} - e^{-i\phi})/2i;\\
&\left|e^{i\phi}\right|^2 = 1;\\
&\cos 2\phi = \cos^2\phi - \sin^2\phi;\ \sin 2\phi = 2\sin\phi\cos\phi
\end{aligned}$$