Notation: ket $|\alpha\rangle$ represents a quantum state without specific basis

Spin-1/2 particle

Basis $|\pm z\rangle$, $|\pm x\rangle$

eigenstates of $\hat{S}_z$ or $\hat{S}_x$

Potential well

Basis - Eigenstates of $\hat{H}$

$\hat{H}|n\rangle = E_n|n\rangle$

Any state can be decomposed in the basis

$|\alpha\rangle = c_+|+z\rangle + c_-|-z\rangle$

$|\alpha\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle + \ldots$

Identity

$\hat{I} = |+z\rangle\langle +z| + |-z\rangle\langle -z|$

$\hat{I} = \sum_n |n\rangle\langle n|$ or $\int_{-\infty}^{+\infty} dx\, |x\rangle\langle x|$

$$|\alpha\rangle = \hat{I}|\alpha\rangle = |+z\rangle \underbrace{\langle +z|\alpha\rangle}_{c_+} + |-z\rangle\underbrace{\langle -z|\alpha\rangle}_{c_-}$$

$$|\alpha\rangle = \hat{I}|\alpha\rangle = \sum_n |n\rangle \underbrace{\langle n|\alpha\rangle}_{c_n}$$

or

$$|\alpha\rangle = \hat{I}|\alpha\rangle = \int_{-\infty}^{+\infty} dx\, \underbrace{\langle x|\alpha\rangle}_{\psi_\alpha(x)} \cdot |x\rangle$$

$$c_n = \langle n|\alpha\rangle = \langle n|\hat{I}|\alpha\rangle = \int_{-\infty}^{+\infty} \langle n|x\rangle\langle x|\alpha\rangle\, dx =$$

$$= \int_{-\infty}^{+\infty} \psi_n^*(x)\,\psi_\alpha(x)\, dx$$

Normalization $\langle\alpha|\alpha\rangle = 1$

$$\langle\alpha|\alpha\rangle = \langle\alpha|+z\rangle\langle +z|\alpha\rangle + \langle\alpha|-z\rangle\langle -z|\alpha\rangle = |c_+|^2 + |c_-|^2 = 1$$

$$\langle\alpha|\alpha\rangle = \sum_n |c_n|^2 = 1$$

or

$$1 = \langle\alpha|\alpha\rangle = \int_{-\infty}^{+\infty} \langle\alpha|x\rangle\langle x|\alpha\rangle\, dx =$$

$$= \int_{-\infty}^{+\infty} \psi_\alpha^*(x)\,\psi_\alpha(x)\, dx = \int_{-\infty}^{+\infty} |\psi_\alpha|^2 dx$$

Transition to a different basis

z-basis $|\alpha\rangle = \langle +z|\alpha\rangle|+z\rangle + \langle -z|\alpha\rangle|-z\rangle$
$|\pm z\rangle = \langle +x|\pm z\rangle|+x\rangle + \langle -x|\pm z\rangle|-x\rangle$

$$\underbrace{\begin{pmatrix} \langle +z|\alpha\rangle \\ \langle +x|\alpha\rangle \end{pmatrix}}_{z\text{-basis}} = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix} \underbrace{\begin{pmatrix} \langle +x|\alpha\rangle \\ \langle -x|\alpha\rangle \end{pmatrix}}_{x\text{-basis}}$$

x-representation: $\langle x|\alpha\rangle = \psi_\alpha(x)$
p-representation $\langle p|\alpha\rangle$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$\langle p|\alpha\rangle = \int_{-\infty}^{+\infty} \langle p|x\rangle\langle x|\alpha\rangle\, dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, \psi_\alpha(x)\, dx$$

Average value of an operator

$\langle \hat{A}\rangle = \langle \alpha|\hat{A}|\alpha\rangle \leftarrow$ use $|\alpha\rangle$ and $\hat{A}$ in the <u>same</u> basis

$$\langle \hat{A}\rangle = (c_+^* \ c_-^*) \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\langle \hat{A}\rangle = \langle \alpha|\hat{I}\hat{A}|\alpha\rangle = \int_{-\infty}^{+\infty} \langle \alpha|x\rangle\langle x|\hat{A}|\alpha\rangle\, dx =$$

$$= \int_{-\infty}^{+\infty} \psi_\alpha^*(x)\, \hat{A}\Big|_{\text{in } x\text{-representation}} \psi_\alpha(x)\, dx$$

Example $\langle x\rangle = \int_{-\infty}^{+\infty} x\, |\psi_\alpha(x)|^2 dx$

$$\langle p\rangle = \int_{-\infty}^{+\infty} \psi_\alpha^*(x) \left(-i\hbar \frac{d}{dx} \psi_\alpha(x)\right) dx$$

Average energy $\langle E\rangle = \langle \alpha|\hat{H}|\alpha\rangle$

If we choose the basis of eigenstates of $\hat{H}$

$|\alpha\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle + \ldots$

$$\langle \alpha|\hat{H}|\alpha\rangle = \langle \alpha|c_1 E_1|1\rangle + \langle \alpha|c_2 E_2|2\rangle + \langle \alpha|c_3 E_3|3\rangle + \ldots$$

$$= c_1 E_1 \underbrace{\langle \alpha|1\rangle}_{c_1^*} + c_2 E_2 \underbrace{\langle \alpha|2\rangle}_{c_2^*} + c_3 E_3 \underbrace{\langle \alpha|3\rangle}_{c_3^*} \ldots =$$

$$= E_1|c_1|^2 + E_2|c_2|^2 + E_3|c_3|^2$$

Time evolution $|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle$

In the basis of eigenstates of the Hamiltonian

$$e^{-i\hat{H}t/\hbar}|n\rangle = e^{-iE_nt/\hbar}|n\rangle$$

$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}\sum_n c_n|n\rangle = \sum_n c_n e^{-i\hat{H}t/\hbar}|n\rangle =$$

$$= \sum_n c_n e^{-iE_nt/\hbar}|n\rangle$$

Time-dependent probability to be in some state $|\beta\rangle$

$$P = |\langle\beta|\alpha(t)\rangle|^2 = \left|\sum_n c_n e^{-iE_nt/\hbar}\langle\beta|n\rangle\right|^2$$

Time-dependent operator average

$$\langle A\rangle(t) = \langle\alpha(t)|\hat{A}|\alpha(t)\rangle$$

Note

$$\langle E\rangle(t) = \langle\alpha(t)|\hat{H}|\alpha(t)\rangle = \langle\alpha(t)|\hat{H}\sum c_n e^{-iE_nt/\hbar}|n\rangle =$$

$$= \sum_n c_n E_n \langle\alpha(t)|n\rangle e^{-iE_nt/\hbar} = \sum_n c_n E_n c_n^* \cancel{e^{iE_nt/\hbar}} \times$$

$$\times \cancel{e^{-iE_nt/\hbar}} = \sum_n E_n |c_n|^2 \quad \text{– no time dependence}$$

Same for any operator for which $\{|n\rangle\}$ is the basis of the eigenstates.