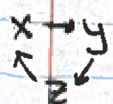
Formal definition of angular momentum/spin operators and states

So far we worked with spin-1/2 particle

$$\hat{S}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle = \hbar\cdot\frac{1}{2}|+z\rangle \; ; \quad \hat{S}_z|-z\rangle = \hbar\left(-\frac{1}{2}\right)|-z\rangle$$

In the eigenbasis of one of the spin components, two others cannot be measured precisely



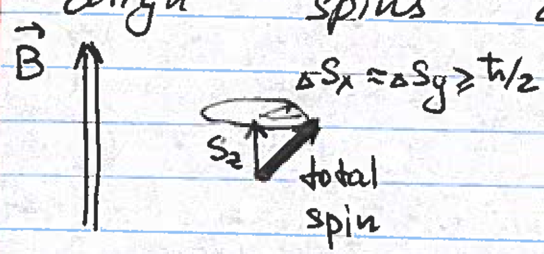
$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \; ; \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \; ; \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

Uncertainty principle $\Delta S_x \cdot \Delta S_y \geq \frac{\hbar}{2}|\langle S_z\rangle|$

If a particle is in $|+z\rangle$ (or $|-z\rangle$ state)

$$\Delta S_x \cdot \Delta S_y \geq \frac{\hbar^2}{4} \qquad \Delta S_x \approx \Delta S_y \geq \frac{\hbar}{2}$$

Because of the uncertainty principle, we fundamentally cannot precisely align spins along one direction



N independent spins

Total average spin $N\cdot\frac{\hbar}{2}$

Total uncertainty (standard deviation)

$$\frac{\text{signal}}{\text{noise}} \propto \frac{N\,\hbar/2}{\sqrt{N}\,\hbar/2} = \sqrt{N}$$

$$\Delta S_{x\,tot} = \sqrt{\sum_{i=1}^{N}\Delta S_x^2} = \sqrt{N}\cdot\Delta S_x = \underline{\underline{\sqrt{N}\,\frac{\hbar}{2}}}$$

spin projection noise

What is the length of the spin?

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hbar^2}{4}\hat{\sigma}_x^2 + \frac{\hbar^2}{4}\hat{\sigma}_y^2 + \frac{\hbar^2}{4}\hat{\sigma}_z^2 = \frac{3}{4}\hbar^2\,\hat{\mathbb{1}}$$

$\hat{S}^2$ commutes with any spin component (since it is an identity)

$$\hat{S}^2|+z\rangle = \frac{3}{4}\hbar^2 \underbrace{\hat{\mathbb{1}}|+z\rangle}_{|+z\rangle} = \hbar^2\,\frac{1}{2}\cdot\frac{3}{2}\,|+z\rangle$$ same for $|-z\rangle$

If we define $s = 1/2$ for spin $1/2$ particle

$$\hat{S}^2|\pm z\rangle = \hbar^2 s(s+1)|\pm z\rangle$$

~~For~~ possible $\hat{S}_z$ outcomes : $\pm s = m = \pm 1/2$

$$\hat{S}_z|\pm z\rangle = \hbar\cdot m\,|\pm z\rangle$$ for $m = \pm 1/2$ correspondingly.

These definitions can be generalized for any value of spin!

We also already used it for orbital angular momentum $\hat{\vec{L}}$ in H-atom description

$$\hat{L}^2\,\psi_{nlm}(\vec{r}) = \hbar^2\,l(l+1)\,\psi_{nlm}(\vec{r})$$

$$\hat{L}_z\,\psi_{nlm}(\vec{r}) = \hbar m\,\psi_{nlm}(\vec{r}) \qquad m = -l, -l+1 \ldots l-1, l$$

We ~~[illegible]~~ are going to use such notation for any angular momentum operators $\longrightarrow$ back to Js!

Formal "names" for $|\pm z\rangle$ states on a spin-½

$$|+z\rangle \equiv |s=\tfrac{1}{2}, m=\tfrac{1}{2}\rangle \quad \left(= |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

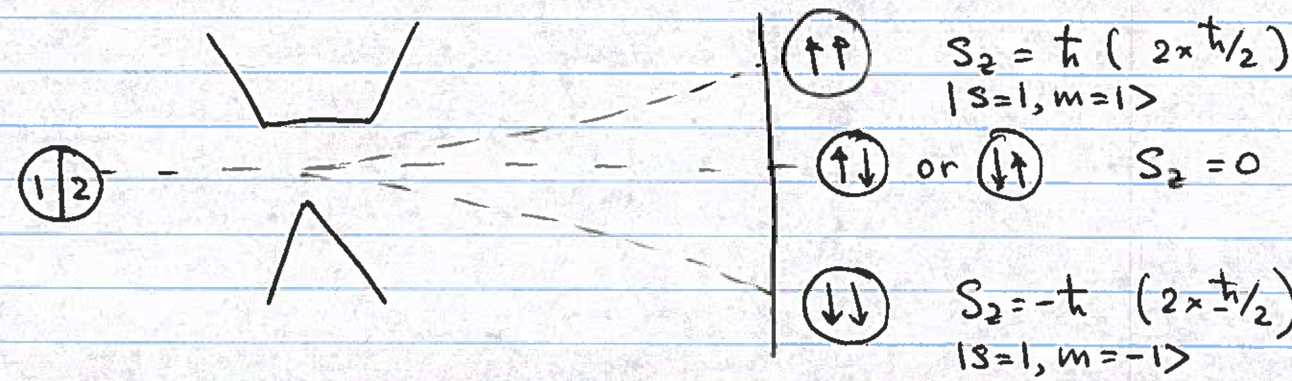
$$|-z\rangle \equiv |s=\tfrac{1}{2}, m=-\tfrac{1}{2}\rangle \quad \left(= |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

Many elementary particles have spin ½ (electrons, protons, neutrons)
Photon is spin -1 (but we can treat its polarization as a spin -½ particle)

Interaction bosons (like W-boson) often are ~~are~~ spin-1, but too short-lived for quantum applications

Most spin 1 (or higher) particles in everyday life → composites.

"Toy" model: two spin ½ particles trapped together. This corresponds to both <u>spin 1</u> and <u>spin 0</u> total spin



Quantum particles that are indistinguishable must produce a state symmetric or ~~anty~~ anti-symmetric to changing order of the particle

$S_z = 0$ → $S=0$ particle $\quad |s=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$S_z = 0$ → $S=1$ particle $\quad |s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

Let's focus on spin 1 particle (switching back to $\hat{\vec{J}}$ notation)

Spin 1 $\quad \hat{J}^2 \, | \text{spin 1 state} \rangle = \hbar^2 \underbrace{1 \cdot 2}_{j(j+1)} \, | \text{spin-1 state} \rangle \qquad j = 1$

$\hat{J}_z$ eigenstates : $|1,1\rangle, \; |1,0\rangle, \; |1,-1\rangle$

$\hat{J}_z \, |\overset{j}{1},\overset{m}{1}\rangle = \overset{m=1}{\hbar} \, |1,1\rangle \qquad \hat{J}_z \, |1,0\rangle = 0 \cdot |1,0\rangle$

$\hat{J}_z \, |1,-1\rangle = -\hbar \, |1,-1\rangle$

Basis vectors : need three.

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{J}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

One can check that

$$\hat{J}^2 = \hat{J}_z^2 + \hat{J}_x^2 + \hat{J}_y^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hbar^2 \underbrace{j(j+1)}_{j=1} \, \mathbb{1}$$

We can use these matrices to find eigenstates of $\hat{J}_x$ or $\underline{\underline{\hat{J}_y}}$ operator.

Full calculations $\longrightarrow$ textbook

$\hat{J}_y|+y\rangle = \hbar|+y\rangle$ $\qquad$ $\hat{J}_y|0_y\rangle = 0.|0_y\rangle$ $\qquad$ $\hat{J}_y|-y\rangle = -\hbar|-y\rangle$

$$|+y\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \hat{J}_y|+y\rangle = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \hbar\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$-{}^{i}/\sqrt{2}\, c_2 = c_1$$

$$i(c_1 - c_3) = c_2 \qquad |+y\rangle = \begin{pmatrix} -{}^{i}/\sqrt{2}\, c_2 \\ c_2 \\ {}^{i}/\sqrt{2}\, c_2 \end{pmatrix}$$

$${}^{i}/\sqrt{2}\, c_2 = c_3$$

Normalization: $\left|-\frac{ic_2}{\sqrt{2}}\right|^2 + |c_2|^2 + \left|\frac{ic_2}{\sqrt{2}}\right|^2 = 1$

$$\left(\frac{1}{2} + 1 + \frac{1}{2}\right)|c_2|^2 = 1$$

$$|c_2|^2 = {}^1/2$$

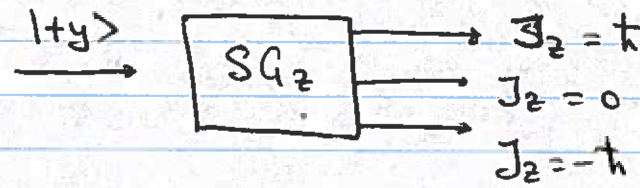
To make the first element c_1 positie and real, we choose $c_2 = {}^{i}/\sqrt{2} \longrightarrow c_1 = {}^1/2,\ c_3 = -{}^1/2$

$$|+y\rangle = \begin{pmatrix} {}^1/2 \\ {}^{i}/\sqrt{2} \\ -{}^1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$$

Similarly

$$|0_y\rangle = \begin{pmatrix} {}^1/\sqrt{2} \\ 0 \\ -{}^1/\sqrt{2} \end{pmatrix} \qquad |-y\rangle = \begin{pmatrix} {}^1/2 \\ -{}^{i}/\sqrt{2} \\ -{}^1/2 \end{pmatrix}$$

If we run a spin 1 particle through a SG_z apparatus, we will find three possible outcomes

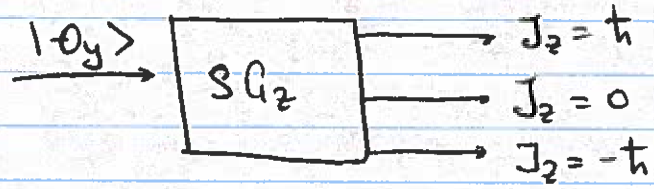


$P_+ = 1/4$

$P_0 = 2/4 = 1/2$

$P_- = 1/4$

same for $|-y\rangle$



$P_+ = 1/2$

$P_0 = 0$

$P_- = 1/2$

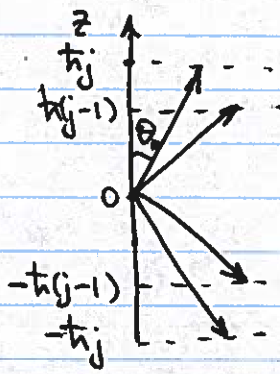
More thoughts about spin orientation

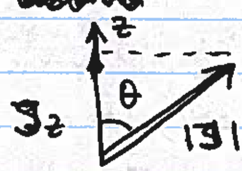
As we discussed, even in a maximally aligned state $|j, m=j\rangle$ $\left(\text{or anti-aligned } |j, m=-j\rangle\right)$ spins are not really aligned with z-axis

"Length" of $\hat{\vec{J}}$. $\longrightarrow \sqrt{\langle \hat{J}^2 \rangle} = |J|$

$$|J| = \sqrt{\hbar^2 j(j+1)} = \hbar\sqrt{j(j+1)}$$

$$J_z = \hbar m \qquad m = -j \dots j$$

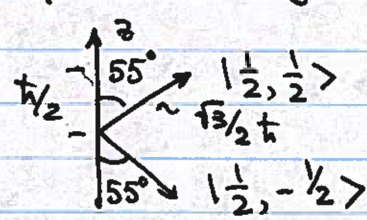




$$\cos\theta_m = \frac{J_{z\,m}}{|J|} = \frac{\hbar m}{\hbar\sqrt{j(j+1)}}$$

"Best" alignment $m=j$ $\cos\theta_{min} = \dfrac{j}{\sqrt{j(j+1)}} = \sqrt{\dfrac{j}{j+1}}$

Spin-1/2 $j = 1/2$ $\cos\theta_{min} = \sqrt{\dfrac{1/2}{3/2}} = \sqrt{1/3}$



$\theta_{min} \approx 55°$

Spin-1 $j=1$ $\cos\theta_{min} = 1/\sqrt{2}$ $\theta_{min} = 45°$