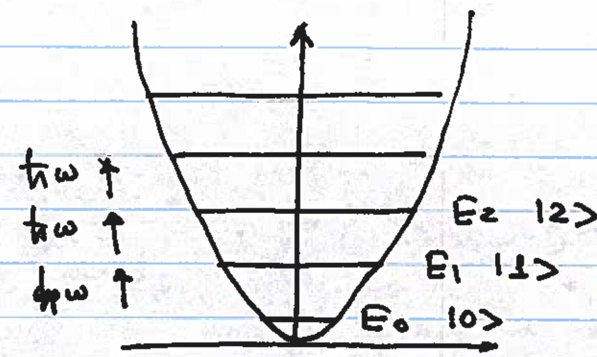
Simple Harmonic Oscillator (SHO)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$



We figured that a particle in this potential has equidistant spectrum

$E_n = \hbar\omega(n+\frac{1}{2})$ $\quad n=0,1\ldots$

and $\psi_n(x)$ are known

Ground-state energy

$E_0 = \frac{1}{2}\hbar\omega$

$$\psi_0(x) = \langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-m\omega x^2/2\hbar}$$

to go from $|0\rangle$ to $|n\rangle$ we need to move up by n quanta of energy (move up and down the energy ladder)

Throw back to $\hat{J}^+$ and $\hat{J}_-$ $\rightarrow$ operators that "moved" the spin states along m-ladder

How to define operators that do that?

Classical energy $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega\left[\frac{p^2}{2m\hbar\omega} + \frac{\omega m x^2}{2\hbar}\right]$

$$= \hbar\omega\underbrace{\left(\sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{ip}{\sqrt{2m\hbar\omega}}\right)}_{a}\underbrace{\left(\sqrt{\frac{m\omega}{2\hbar}}\,x - \frac{ip}{\sqrt{2m\hbar\omega}}\right)}_{a^*}$$

We have to be more careful with the quantum case since $[\hat{x},\hat{p}] = i\hbar$

Ladder operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$[\hat{a},\hat{a}^+] = \frac{m\omega}{2\hbar}\left[\left(\hat{x} + \frac{i}{\omega m}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) - \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(x + \frac{i}{m\omega}\hat{p}\right)\right]$$

$$= \frac{m\omega}{2\hbar}\left[\frac{i}{m\omega}\hat{p}\hat{x} - \frac{i}{m\omega}\hat{x}\hat{p} - \frac{i}{m\omega}\hat{x}\hat{p} + \frac{i}{m\omega}\hat{p}\hat{x}\right] =$$

$$= \frac{m\omega}{2\hbar} \cdot \frac{2i}{m\omega}\underbrace{[\hat{p},\hat{x}]}_{= -[\hat{x},\hat{p}] = -i\hbar} = 1$$

$$[\hat{a},\hat{a}^+] = 1 \qquad \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1 \qquad \underline{\hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+) \qquad \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^+)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{\hbar\omega}{4}(\hat{a} - \hat{a}^+)^2 + \frac{\hbar\omega}{4}(\hat{a} + \hat{a}^+)^2$$

$$= -\frac{\hbar\omega}{4}\left[\hat{a}^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^{+2}\right] + \frac{\hbar\omega}{4}\left[\hat{a}^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^{+2}\right]$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right) = \frac{\hbar\omega}{2}\left(2\hat{a}^+\hat{a} + 1\right) = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) \qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Number operator $\hat{n} = \hat{a}^+\hat{a}$ $\qquad \hat{n}|n\rangle = n|n\rangle$

This operator "counts" the number of energy quanta needed to get to the state $|n\rangle$ from the ground state $|0\rangle$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \text{lowering operator}$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \text{raising operator}$$

$$\underbrace{\hat{a}^+\hat{a}}_{\hat{n}}|n\rangle = \hat{a}^+\left[\sqrt{n}\,|n-1\rangle\right] = n|n\rangle$$

We can use this operators to find the wave functions for excited states

In x- representation $\hat{a} \rightarrow \sqrt{\frac{\omega m}{2\hbar}} \left(x + \frac{\hbar}{m\omega} \frac{\partial}{\partial x}\right)$

$\hat{a}|0\rangle = 0 \qquad \hat{a}\,\psi_0(x) = 0$

we can find $\psi_0(x)$ by solving

$$\frac{\hbar}{m\omega} \frac{d\psi_0}{dx} + x\,\psi_0(x) = 0 \quad \Rightarrow \psi_0 = A e^{-\frac{m\omega x^2}{2\hbar}}$$

$$|A|^2 \int_{-\infty}^{+\infty} e^{-m\omega x^2/2\hbar}\, dx = 1 \Rightarrow A = \sqrt{\frac{m\omega}{\pi\hbar}}$$

$$|1\rangle = \hat{a}^+|0\rangle \qquad |2\rangle = \frac{1}{\sqrt{2}}(\hat{a}^+)^2|0\rangle \;.. \qquad |n\rangle = \frac{(\hat{a}^+)^n|0\rangle}{\sqrt{n(n-1)\ldots 2}}$$

Now we have two complementary ways of calculating properties of SHO

If we need to find anything related to the position of the ~~operator~~ particle, we can use known wave functions $\{\psi_n(x)\}$

Alternatively, we can use energy quantisation and ladder operators

↳ we are circling back to the foundational theory of Plank

Electromagnetic radiation as SHO

Classical EM field, Maxwell's equations in vacuum

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \vec{E} = 0 \qquad \nabla \vec{B} = 0$$

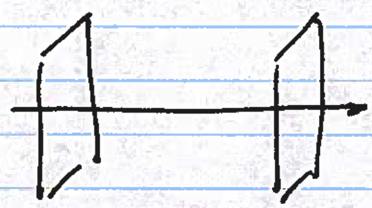
Quantum description $\hat{\vec{E}}, \hat{\vec{B}}$, they don't commute

Energy of EM field

Classical: Energy $\frac{1}{2}\int \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) dV$

Quantum: Hamiltonian $\hat{H} = \frac{1}{2}\int \left(\varepsilon_0 \hat{E}^2 + \frac{1}{2\mu_0}\hat{B}^2\right) dV$

Simplest case: plane e-m wave b/w flat mirrors



$\vec{k} = (0, 0, k_z)$ light travels along z

$\vec{E} = (E, 0, 0)$

$\vec{B} = (0, B, 0)$

$$\frac{\partial B}{\partial z} = \frac{1}{c}\frac{\partial E}{\partial t}$$

Wave equation $\frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$ $\quad E(0) = E(L) = 0$

Solution

$$\vec{E}(z,t) = \underbrace{E(t)}_{\text{amplitude}} \sin kz = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q(t) \sin kz$$

$$B(z,t) = B_0 \cos kz = -\sqrt{\frac{2}{c^2 \varepsilon_0 V}} \cos kz \, \dot{q}(t)$$

In quantum case $q(t) \to \hat{q}(t)$ $\quad \dot{q}(t) \to \hat{\dot{q}}(t)$

Hamiltonian

$$\hat{H} = \frac{1}{2}\int_V dV\left(\varepsilon_0 \hat{E}^2 + \frac{1}{\mu_0}\hat{B}^2\right) =$$

$$= \frac{1}{2} S \int_0^L \left(\varepsilon_0 \frac{2\omega^2}{V\varepsilon_0}\hat{q}^2 \sin^2 kz + \frac{1}{\mu_0}\frac{2}{c^2 V^2 \varepsilon_0}\hat{\dot{q}}^2 \cos^2 kz\right) dz$$

$$= \frac{1}{2}\hat{\dot{q}}^2 + \frac{1}{2}\omega^2 \hat{q}^2 \qquad \hat{\dot{q}} \longleftrightarrow \hat{p}, \quad \hat{q} \longleftrightarrow \hat{x}$$

looks like $\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 \longrightarrow$ SHO !

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} - i\hat{p})$$

$|n\rangle$ — ~~number~~ an e-m wave with n photons (n quanta of energy)

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)|n\rangle = n\hbar\omega + \frac{\hbar\omega}{2}$$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ annihilation operator

$\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$ creation operator

$$\hat{E}(t,z) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat{a}^+ e^{-i\omega t} + \hat{a}e^{i\omega t}\right)\sin kz$$

Wave-particle duality

Electric field operator of a plane wave has all wave properties, but the amount of energy the wave carries can only change by fixed quantum $\hbar\omega$, just like for a particle.

$a^{\dagger}|$  $\rangle = |$  $\rangle$