Extra "family members" – raising and lowering operators

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad \hat{J}_- = \hat{J}_x - i\hat{J}_y$$

Spin - ½ particle

$$\hat{J}_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

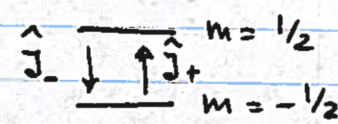
$$\hat{J}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_+|½,½\rangle = \hbar\cancel{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\hat{J}_-|½,½\rangle = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\hat{J}_+|½,-½\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\hat{J}_-|\tfrac{1}{2},-½\rangle = 0$$



Spin - 1 particle

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} =$$

$$= \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{J}_+|1,-1\rangle = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\hbar\,|1,0\rangle$$

$\hat{J}_+|1,0\rangle = \sqrt{2}\hbar\,|1,1\rangle \qquad \hat{J}_+|1,1\rangle = 0$

Similarly

$\hat{J}_-|1,1\rangle = \sqrt{2}\hbar\,|1,0\rangle \qquad \hat{J}_-|1,0\rangle = \sqrt{2}\hbar\,|1,-1\rangle$ & $\hat{J}_-|1,-1\rangle = 0$

$$\begin{array}{cl} \text{———} & m=1 \\ \hat{J}_-\downarrow \quad \uparrow\hat{J}_+ & \\ \text{———} & m=0 \\ \hat{J}_-\downarrow \quad \uparrow\hat{J}_+ & \\ \text{———} & m=-1 \end{array}$$

$\hat{J}_+$ "raises" the state on the m ladder

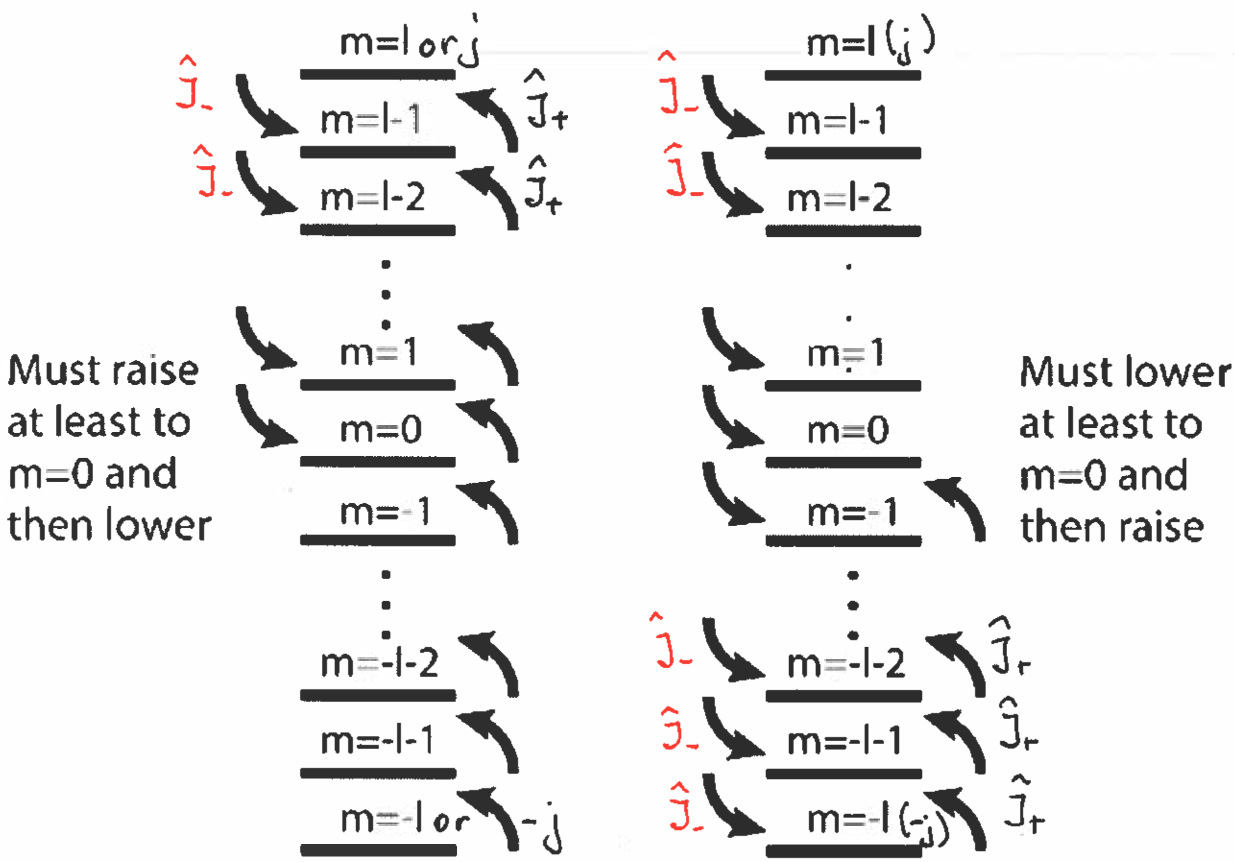
$\hat{J}_-$ "lowers" the state on the m ladder

$\hat{J}_+$ & $\hat{J}_-$ also known as ladder operators

In general $\quad \hat{J}_+|j,m\rangle \propto |j,m+1\rangle$

$\hat{J}_-|j,m\rangle \propto |j,m-1\rangle$

$$2j+1 \text{ states} \left\{ \begin{array}{ll} m=j & \text{———} \\ m=j-1 & \text{———} \\ m=j-2 & \text{———} \\ \vdots & \text{———} \\ & \dots \\ m=-j & \text{———} \end{array} \right. \qquad \begin{array}{cc} \uparrow \cdots & \downarrow \cdots \\ \uparrow \cdots & \downarrow \cdots \\ \uparrow \hat{J}_+ & \downarrow \hat{J}_- \end{array}$$



First form of the exponential disentangling operator identity

Second form of the exponential disentangling operator identity