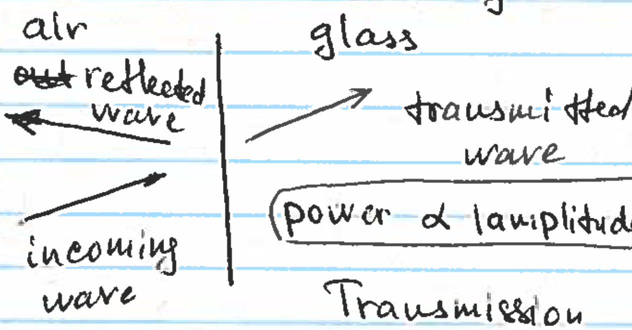
Particle transmission & reflection from a potential step

Classical analogue air-glass interface



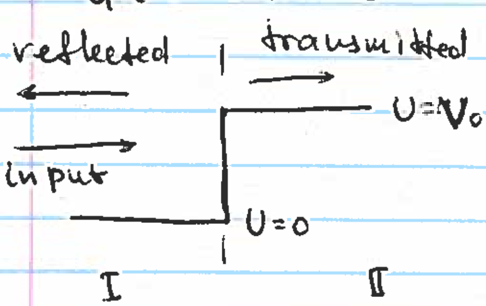
$\boxed{\text{power} \propto |\text{amplitude}|^2}$

Reflection coefficient

$$R = \frac{\text{~~out~~ reflected power}}{\text{input power}}$$

Transmission coefficient $T = 1 - R$

Quantum case



In region I (input + reflected)

$$\psi_I = \underbrace{A e^{ip_0 x/\hbar}}_{\text{input}} + \underbrace{B e^{-ip_0 x/\hbar}}_{\text{reflected}}$$

In region II (transmitted)

$$\psi_{II} = C e^{ip_1 x/\hbar}$$

Particle total energy $E = \dfrac{p_0^2}{2m} = \dfrac{p_1^2}{2m} + V_0$

$$p_0 = \sqrt{2mE} \quad \text{and} \quad p_1 = \sqrt{2m(E - V_0)}$$

Boundary conditions $\psi_I(0) = \psi_{II}(0)$
$\psi_I'(0) = \psi_{II}'(0)$

$$\begin{cases} A + B = C \\ ip_0 \cdot A - ip_0 B = ip_1 \cdot C \end{cases} \Rightarrow p_0(A - B) = p_1(A + B)$$

$$\Rightarrow B = \frac{p_0 - p_1}{p_0 + p_1} A$$

A, B, C - amplitudes of the probability waves

Reflection coefficient $R = \left|\dfrac{B}{A}\right|^2 = \left(\dfrac{p_0 - p_1}{p_0 + p_1}\right)^2$

Interestingly enough, this expression matches perfectly the classical reflection coefficient from a transparent medium

$$\lambda_0 = \frac{2\pi\hbar}{p_0} \qquad \lambda_1 = \frac{2\pi\hbar}{p_1} \quad \Rightarrow \quad R = \left(\frac{\lambda_1 - \lambda_0}{\lambda_1 + \lambda_0}\right)^2$$

For glass $\lambda_{glass} \approx 1.5\,\lambda_{air}$

$$R = \left(\frac{0.5}{2.5}\right)^2 \approx \frac{1}{25} = 4\%$$

But what if $V_0 > E$! Classically, particle cannot exist in this region (classically forbidden region) $\underline{p^2/2m = E - V_0 < 0}$

Quantum particle can, even though with vanishing probability

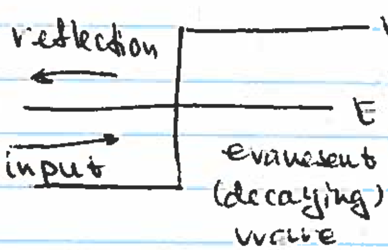
Shrodingre equation $\qquad \dfrac{\hat{p}^2}{2m}\psi(x) + V_0\psi(x) = E\psi(x)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\,\psi(x)$$

$$\frac{d^2\psi}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi = q^2\cdot\psi \qquad q = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

Solutions $\psi(x) = e^{\pm q\cdot x}$

I $\qquad$ II $\qquad$ evanescent wave



$$\text{I}: \quad \psi_I(x) = A e^{ip_0 x/\hbar} + B e^{-ip_0 x/\hbar}$$

$$\text{II}: \quad \psi_{II} = \underline{\underline{C e^{-q\cdot x}}}$$

only one exponent so that $\psi_{II}(x \to \infty) \to 0$

Boundary conditions: $\psi_I(0) = \psi_{II}(0)$ $\quad A + B = C$

$\psi_I'(0) = \psi_{II}'(0)$ $\quad i p_0 A - i p_0 B = -q C$

$$i \hbar p_0 (A - B) = -q (A + B) \qquad B = \frac{i p_0/\hbar + q}{i p_0/\hbar - q} A = \frac{i p_0 + \hbar q}{i p_0 - \hbar q} A$$

$$R = \left| \frac{B}{A} \right|^2 = 1 \qquad \text{but} \qquad B = A e^{i\theta}$$

total reflection

$$B = \frac{(i p_0 + \hbar q)^2}{p_0^2 + (\hbar q)^2} A =$$

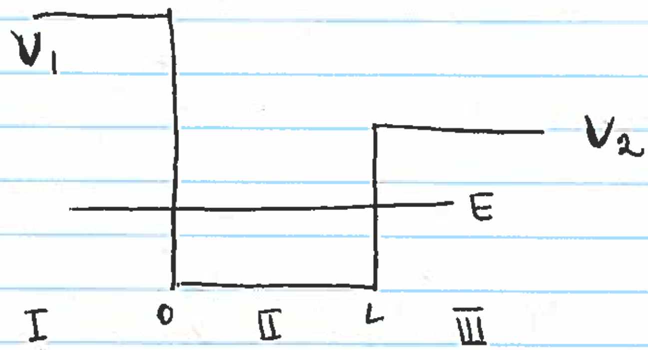
$$= \frac{(\hbar^2 q^2 - p_0^2) + i p_0 \hbar q}{p_0^2 + (\hbar q)^2} A = (\cos\theta + i \sin\theta) A$$

$$\sin\theta = \frac{\hbar q \cdot p_0}{p_0^2 + (\hbar q)^2} = \frac{\sqrt{2m(V_0 - E)}\,\sqrt{2mE}}{2mE + 2m(V_0 - E)} =$$

$$= \frac{\sqrt{(V_0 - E)E}}{V_0}$$

Reflected wave aquire a phase that depends on the parrier height.

Finite potential well



Regions I & III
classically forbidden
(evanescent waves)
Region II
classically allowed
(standing wave)

$$\psi_{III} = A e^{-q_2 x} \qquad q_2 = \sqrt{2m(V_2 - E)/\hbar^2}$$

$$\psi_{II} = C_1 e^{ip_0 x/\hbar} + C_2 e^{-ip_0 x/\hbar} \quad \text{or} \quad C_3 \cos\frac{p_0 x}{\hbar} + C_4 \sin\frac{p_0 x}{\hbar}$$

$$p_0 = \sqrt{2mE} = \hbar k$$

$$\psi_I = B e^{q_1 x} \qquad q_1 = \sqrt{2m(V_1 - E)/\hbar^2}$$

positive exponent, so that $\psi_I(x) \xrightarrow{x \to -\infty} 0$

To find the ~~boundary~~ ~~con~~ energies, corresponding to the stationary states, need to apply the boundary conditions

$$\psi_I(0) = \psi_{II}(0) \qquad \psi_{II}(L) = \psi_{III}(L)$$
$$\psi_I'(0) = \psi_{II}'(0) \qquad \psi_{II}'(L) = \psi_{III}'(L)$$

These equations usually have to be solved numerically.

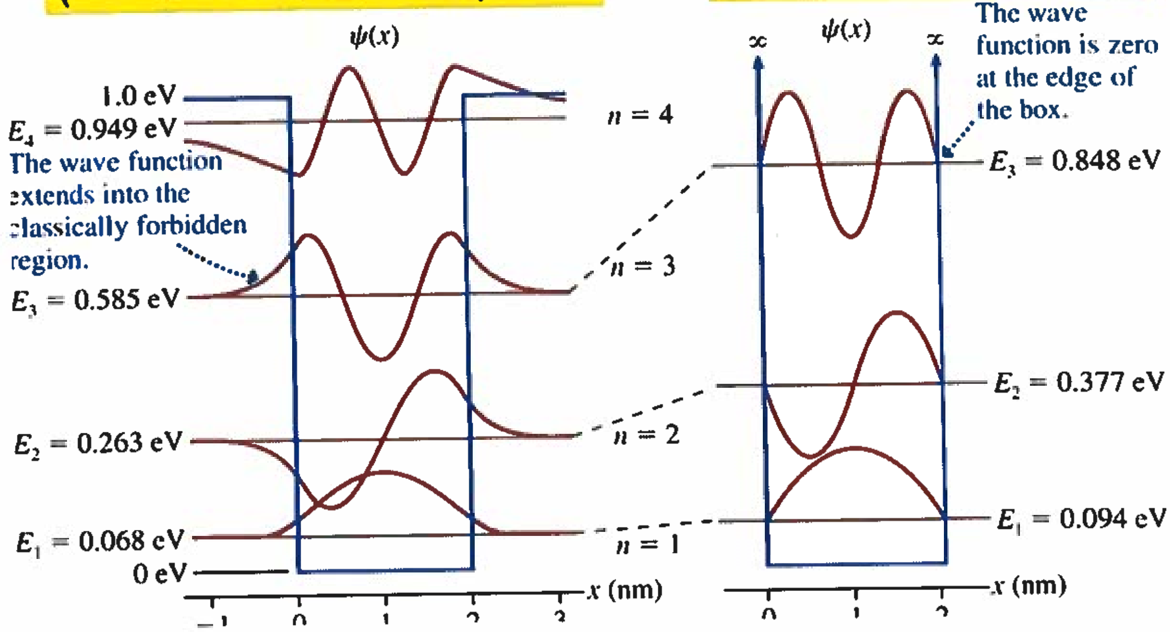
Compare to the infinite well, particle "turning points" are smeared, especially for higher-energy states.

Usually, there is a finite # of possible states.

Comparison of infinite and finite potential wells

Electron in finite square well (*a*=2 nm and V=1.0 eV)

Infinite potential well (*a* = 2 nm and V = ∞)



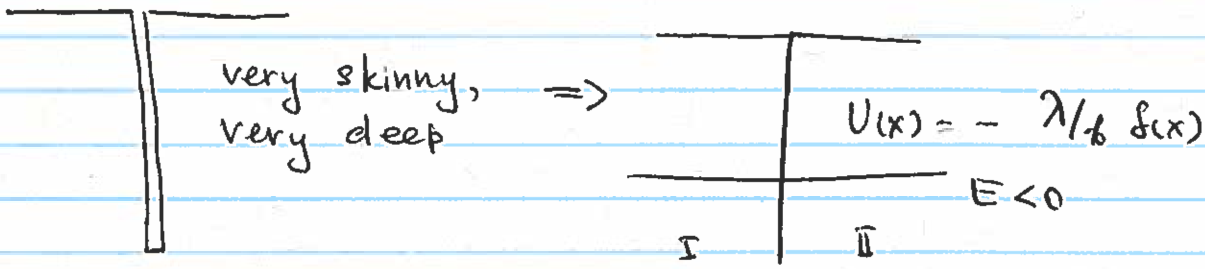
$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2$$

in numbers

$$E_n = \frac{(\hbar c)^2 \pi^2 n^2}{2(mc^2) L^2} = \frac{(197\ eV\cdot nm)^2 \pi^2 n^2}{2 \cdot 0.51 \cdot 10^6 eV \cdot (2nm)^2}$$

$$= 0.094\, eV \cdot n^2$$

Extreme case of a potential well is δ- function well



Both regions I & II are classically forbidden

$$\psi_I(x) = A e^{+qx} \qquad \psi_{II}(x) = B e^{-qx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -|E|\,\psi(x) \qquad \frac{d^2\psi}{dx^2} = q^2\psi \qquad q^2 = \frac{2m|E|}{\hbar^2}$$

Boundary conditions $\psi_I(0) = \psi_{II}(0) \Rightarrow A = B$

$$\psi'(x=0+) - \psi'(x=0-) = -\frac{\lambda}{b}\psi(0)$$

$$-qA - q\cdot A = -\lambda/b \cdot A \qquad \Rightarrow 2q = \lambda/b$$

$$2\cdot\frac{2m|E|}{\hbar^2} = \lambda/b$$

$$E = -\frac{\hbar^2}{4m}\,\lambda/b$$

single energy level