Spherical infinite potential well

Spherical shell incide which particles are free to move

Quite good model for an atomic nucleus : since strong force grows rapidly with distance, particles are essentually free when close, but the attractive interaction "turns on" very quickly after some distance. Thus each individual particle is "trapped" in a deep potential well created by other particles, and it can be approximately be considered spherically symmetric

$$V(r) = \begin{cases} 0 & r<a \\ \infty & r>a \end{cases} \qquad \hat{H} = \frac{\hat{p}^2}{2m} \quad r<a$$

free motion

$$\psi(r) = R_{E,l}(r)\, Y_{lm}(\theta,\varphi)$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + \overbrace{\frac{2mE}{\hbar^2}}^{k^2} R = 0$$

Ground state $l=0$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + k^2R = 0 \xrightarrow{R=u/r} \frac{d^2u}{dr^2} + k^2u = 0$$

(looks familiar)

$$R(r) = A\frac{\cos kr}{r} + B\frac{\sin kr}{r}$$

Boundary condition $R(0)=0 \Rightarrow A=0$

$R(r=a)=0 \qquad ka = \pi n$

$$E_{n,l=0} = \frac{\pi^2 n^2 \hbar^2}{2ma^2} \qquad R_{n,l=0} = \frac{\sin \pi n r/a}{r}$$

Things are getting more complex for $l>0$

if we rewrite the ~~sk~~ Schrodinger eqn in the dimentionless units $\rho = kr$

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho} + \left[1 - \frac{l(l+1)}{\rho}\right] R = 0$$

Solutions: spherical Bessel functions $j_l(\rho)$

$$j_l(\rho) = (-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\sin\rho}{\rho}$$

and spherical Neumann functions $n_l(\rho)$

$$n_l(\rho) = -(-\rho)^l \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^l \frac{\cos\rho}{\rho}$$

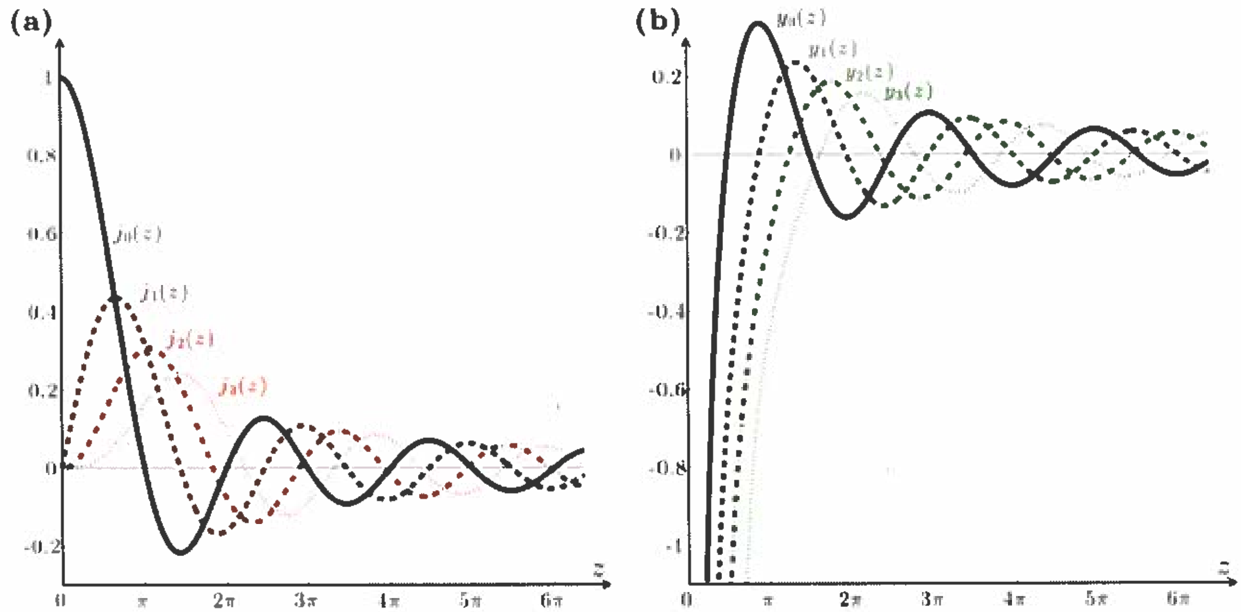
(ugly cousins of $\sin x$ & $\cos x$ and regular Bessel $J_l(x)$ and Neumann $N_l(x)$ functions. The latters are always there for 2d problems)

~~Bd~~ Boundary conditions $R(\rho=0)=0 \rightarrow$ only $j_l(\rho)$ allowed

$R_l(\rho = ka) = 0$ $\quad j_l(ka) = 0$

Zeros of bessel functions are known
(I'll label them $z_n^{(l)} \rightarrow j_l(z_n^{(l)}) = 0$)

$$k_n a = z_n^{(l)} \qquad k_n = \frac{z_n^{(l)}}{a} \qquad E_{n,l} = \frac{\hbar^2 (z_n^{(l)})^2}{2ma^2}$$



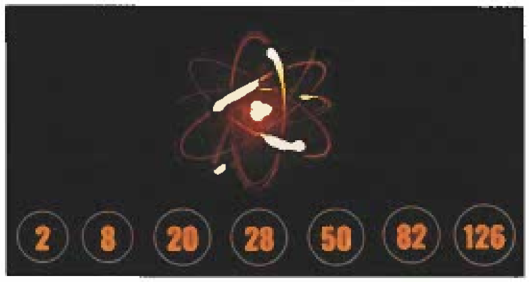
$$j_0(x) = \frac{\sin x}{x}.$$

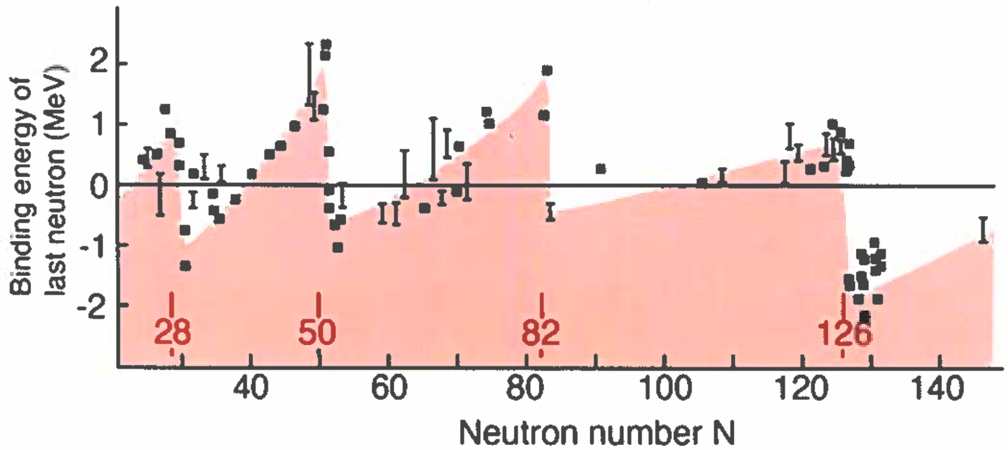
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},$$

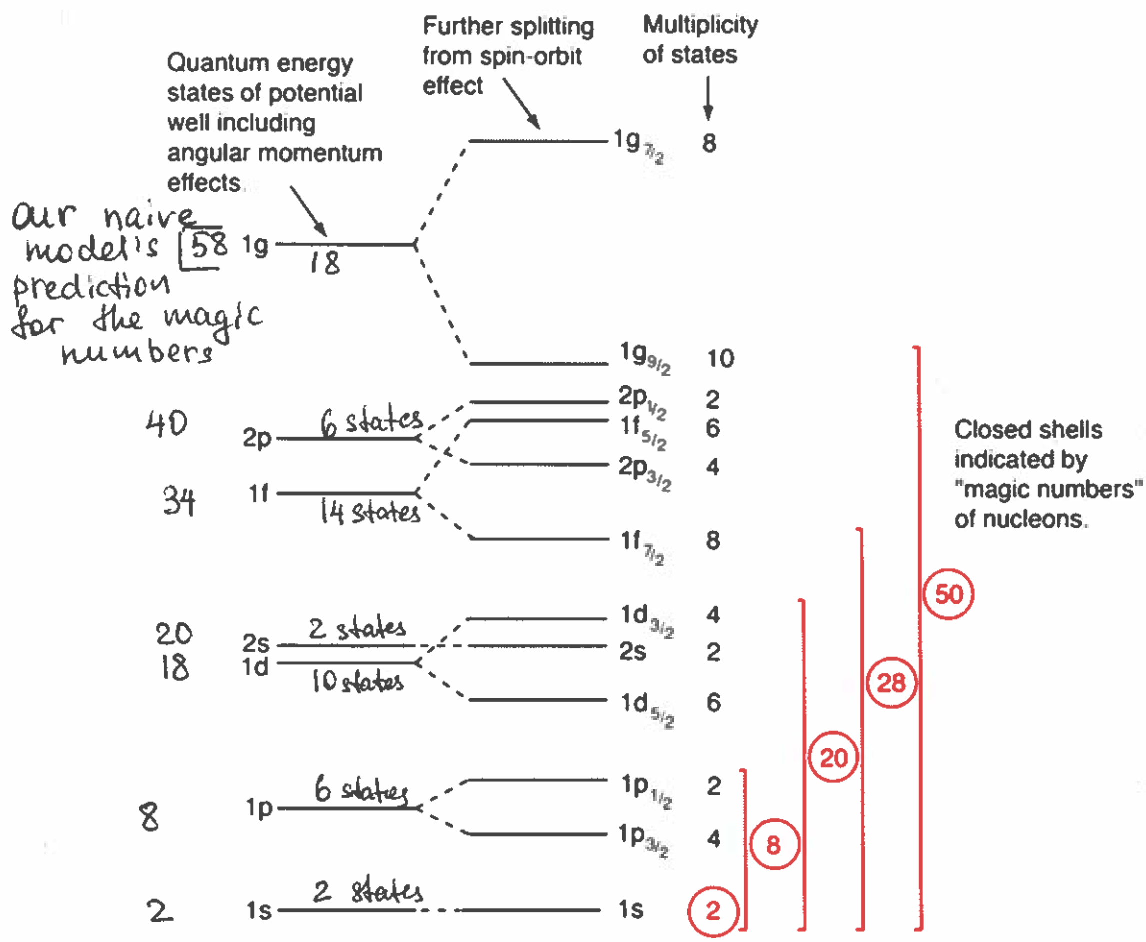
$$j_2(x) = \left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2},$$

$$j_3(x) = \left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\sin x}{x} - \left(\frac{15}{x^2} - 1\right)\frac{\cos x}{x}$$

Number of zero; n	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$
1	3.14159	4.49341	5.76346	6.98793	8.18256
2	6.28319	7.72525	9.09501	10.4171	11.7049
3	9.42478	10.9041	12.3229	13.6980	15.0397
4	12.5664	14.0662	15.5146	16.9236	18.3013
5	15.7080	17.2208	18.6890	20.1218	21.5254


2
8
20
28
50
82
126


Binding energy of last neutron (MeV)
2
1
0
-1
-2
28
50
82
126
40
60
80
100
120
140
Neutron number N


Quantum energy states of potential well including angular momentum effects
Further splitting from spin-orbit effect
Multiplicity of states
our naive model's prediction for the magic numbers
58
1g
18
40
2p
6 states
34
1f
14 states
20
2s
2 states
18
1d
10 states
8
1p
6 states
2
1s
2 states
1g 7/2 8
1g 9/2 10
2p 1/2 2
1f 5/2 6
2p 3/2 4
1f 7/2 8
1d 3/2 4
2s 2
1d 5/2 6
1p 1/2 2
1p 3/2 4
1s
2
8
20
28
50
Closed shells indicated by "magic numbers" of nucleons.