Quantum state vectors (cont)

What we learned so far:

- Quantum state notation $|\alpha\rangle$ ket $\leftrightarrow$ $\langle\alpha|$ bra

* $|\alpha\rangle = c_+|+z\rangle + c_-|-z\rangle$ decomposition in
  $\langle\alpha| = c_+^*\langle+z| + c_-^*\langle-z|$ $|\pm z\rangle$ basis

$P_\pm = |c_\pm|^2$ – probability for a particle in the state $|\alpha\rangle$ to be found in $|\pm z\rangle$ state if S_z is measured

$c_\pm = \langle\pm z|\alpha\rangle$ inner product

probability amplitude

generally, a <u>complex</u> <u>number</u>

$P_\pm = |c_\pm|^2$ – absolute value squared

$|c_\pm|^2 = c_\pm^* c_\pm$ – real non-negative number b/w 0 and 1

$P_\pm = |\langle\pm z|\alpha\rangle|^2$

$$\boxed{\begin{array}{l} z = x+iy \\ z^* = x-iy \\ (e^{i\varphi})^* = e^{-i\varphi} \end{array}}$$

Two quantum states are orthogonal, if $\langle\alpha|\beta\rangle=0$
(i.e if a particle is in one, it cannot be found in another)

And to be physically meaningful, any quantum state must be normalized
$\langle\alpha|\alpha\rangle = 1$

Spin basis: $|+z\rangle$ and $|-z\rangle$ are orthogonal
$\langle+z|-z\rangle = \langle-z|+z\rangle = 0$

In general: $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$ (complex conjugate)

Connection to classical measurements

If a classical spin is oriented in $+z$ direction, we know that its x and y components are zero.

~~How it translates to qua~~
Does it conflicts with quantum world?
No $\longrightarrow$ for classical limit we need to find the average values of S_x and S_y

If a ~~q~~ particle is in a state $|+z\rangle$

$$|+z\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle$$

$$P(S_x = \tfrac{\hbar}{2}) = \tfrac{1}{2} \qquad P(S_x = -\tfrac{\hbar}{2}) = +\tfrac{1}{2}$$

$$\langle S_x \rangle = \frac{\hbar}{2}\cdot\frac{1}{2} + \left(-\frac{\hbar}{2}\right)\cdot\frac{1}{2} = 0$$

matches classical expectation

The average value (or expectation value) of a spin (e.g. S_x) in a state $|\alpha\rangle$

$|\alpha\rangle = C_+|+x\rangle + C_-|-x\rangle$ is

$$\langle S_x \rangle_{|\alpha\rangle} = \left(\frac{\hbar}{2}\right)\cdot|C_+|^2 + \left(-\frac{\hbar}{2}\right)|C_-|^2$$

If we have a larger clump of spins, each of which is randomly oriented, and we send it through SG apparatus, roughly half of them will pull up, another half will pull down, so overall displacement will be close to zero. But not exactly zero, because $N_+ \neq N_-$ in general

For each measurement there will be a small dispalance $\longrightarrow$ uncertainty.

The uncertainty is characterized by a standard deviation:

$$(\Delta S_x)^2 = \langle (S_x - \langle S_x \rangle)^2 \rangle =$$

$$= \sum \begin{pmatrix} \text{probability of} \\ \text{an outcom} \end{pmatrix} \times \begin{pmatrix} \text{possible outcome} - \text{average} \end{pmatrix}^2$$

For example if a particle is in a state $|+z\rangle$: $\langle S_x \rangle = 0$

$$(\Delta S_x)^2_{|+z\rangle} = \frac{1}{2}\left[\left(\frac{\hbar}{2} - 0\right)^2\right] + \frac{1}{2}\left[\left(-\frac{\hbar}{2} - 0\right)^2\right] = \frac{\hbar^2}{4}$$

However, since $|+z\rangle$ is an eigenstate of S_z, we know its value with no uncertaity

$$\langle S_z \rangle = \frac{\hbar}{2} \qquad |+z\rangle = 1 \cdot |+z\rangle + 0 \cdot |-z\rangle$$

$$(\Delta S_z)^2_{|+z\rangle} = 1 \cdot \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)^2 + 0\left(-\frac{\hbar}{2} - \frac{\hbar}{2}\right)^2 = 0$$

If a particle is in a state

$$|\alpha\rangle = c_+ |+z\rangle + c_- |-z\rangle$$

$$\langle S_z \rangle = \left(\frac{\hbar}{2}\right)|c_+|^2 + \left(-\frac{\hbar}{2}\right)|c_-|^2 = \frac{\hbar}{2}\left(|c_+|^2 - |c_-|^2\right)$$

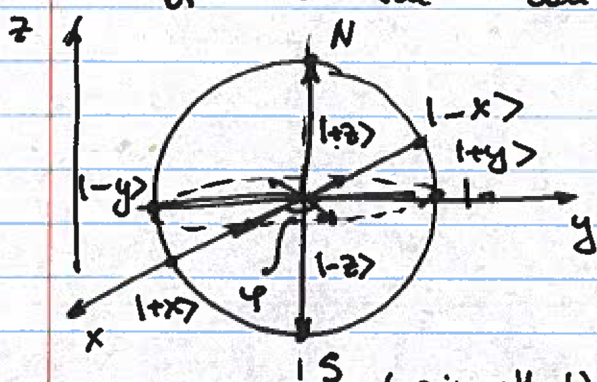
$$(\Delta S_z)^2 = \left(\frac{\hbar}{2} - \langle S_z \rangle\right)^2 \cdot |c_+|^2 + \left(-\frac{\hbar}{2} - \langle S_z \rangle\right)^2 \cdot |c_-|^2$$

Visualization of a spin vector
Bloch sphere

~~When~~ Statement: for a spin we can know only its length and one of the component (typically S_z)

Thus, all possible ends of a spin vector must "live" on a surface of a sphere.

Now, the ~~longitude~~ latitude is given by how close you are to the North $|+z\rangle$ or to the South $|-z\rangle$ poles



Equator is equally distant from both poles, so any spin direction in x−y plane will be equal combination of $|+z\rangle$ and $|-z\rangle$ and different only by ~~that~~ longitude

φ – longitude ~~and~~ (azimuthal) angle

$$\boxed{\left| \begin{matrix}\text{a state in} \\ x-y \text{ plane}\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}} e^{i\varphi}|-z\rangle}$$

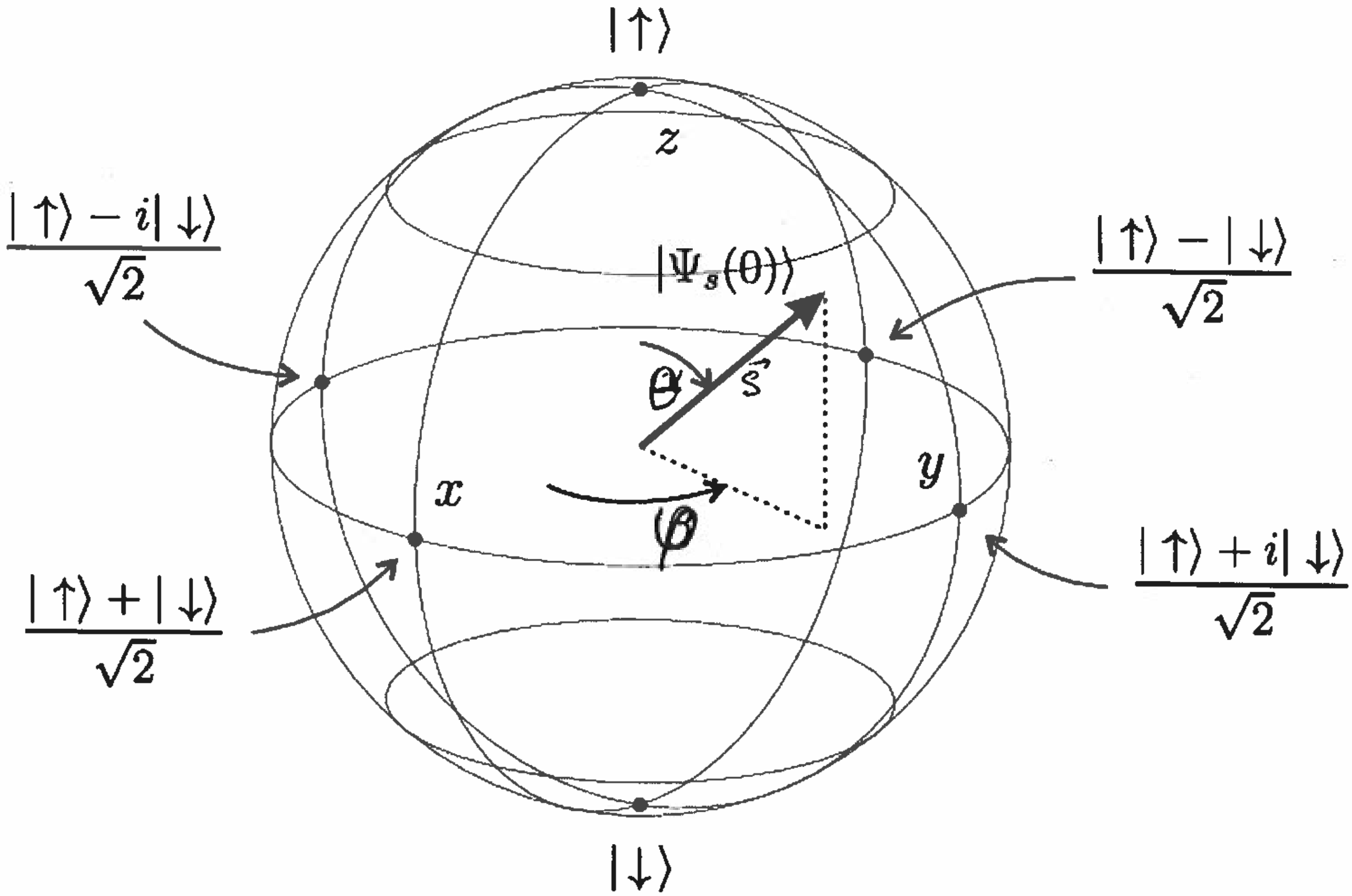
We choose $|+x\rangle$ state to correspond to azimuthal angle $\varphi = 0$

$\varphi = 0$ $\quad e^{i\cdot 0} = 1$ $\quad |+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

$\varphi = \pi$ $\quad e^{i\pi} = \cos\pi + i\sin\pi = -1$ $\quad |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$

$\varphi = \frac{\pi}{2}$ $\quad e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$ $\quad |+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$

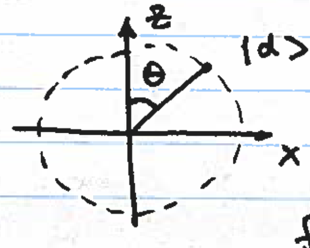
$\varphi = \frac{3\pi}{2}$ $\quad e^{i3\pi/2} = -i$ $\quad |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$



$$|\psi_{\vec{n}}\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}\,|\downarrow\rangle$$

if $\vec{S}$ is oriented along $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

A spin in the x-z plane



$\theta = 0 \qquad |\alpha\rangle = |+z\rangle$

$\theta = \pi \qquad |\alpha\rangle = |-z\rangle$

$\theta = \pi/2 \qquad |\alpha\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$

for any θ

$$|\alpha\rangle = \cos\frac{\theta}{2}|+z\rangle + \sin\frac{\theta}{2}|-z\rangle$$

Any spin orientation (spherical angles θ, φ), if a spin is along

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$|\vec{n}\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$$

$$P_+ = |\langle +z|\vec{n}\rangle|^2 = |\cos\frac{\theta}{2}|^2 = \cos^2\frac{\theta}{2}$$

$$P_- = |\langle -z|\vec{n}\rangle|^2 = |e^{i\varphi}|^2 |\sin\frac{\theta}{2}|^2 = \sin^2\frac{\theta}{2}$$

as we saw in our "experiment" with rotating secong SG apparatus

For $-\vec{n}$ $\qquad \theta \rightarrow \pi - \theta \qquad \cos(\frac{\pi}{2} - \frac{\theta}{2}) = \sin\frac{\theta}{2}$

$\sin(\frac{\pi}{2} - \frac{\theta}{2}) = \cos\frac{\theta}{2}$

$\varphi \rightarrow \pi + \varphi \qquad e^{i(\pi+\varphi)} = e^{i\pi}e^{i\varphi} = -e^{i\varphi}$

$$|\vec{n}\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$$

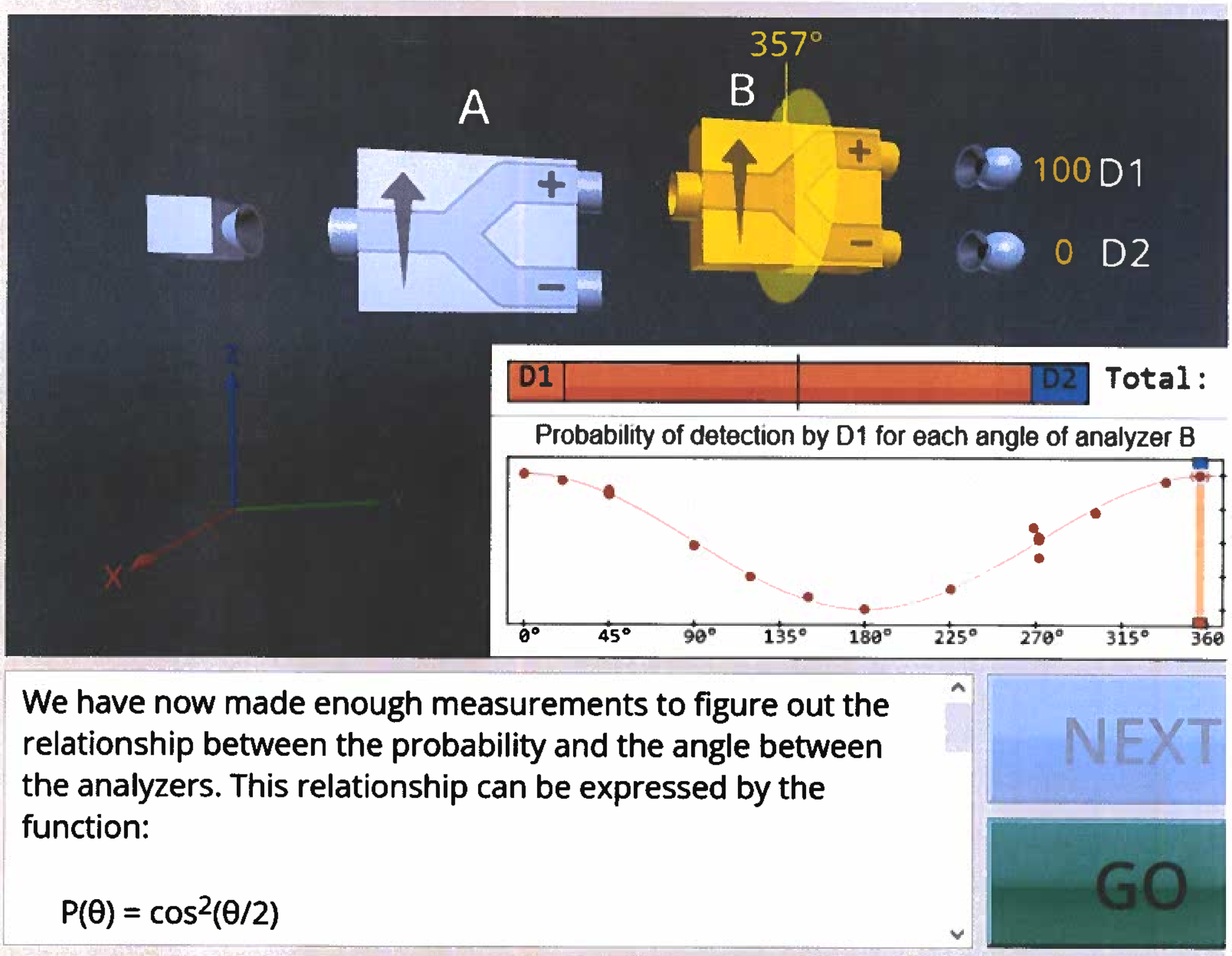
$$|-\vec{n}\rangle = \sin\frac{\theta}{2}|+z\rangle - e^{i\varphi}\cos\frac{\theta}{2}|-z\rangle$$

$$\langle n|-\vec{n}\rangle = \left(\cos\frac{\theta}{2}\langle +z| + e^{-i\varphi}\sin\frac{\theta}{2}\langle -z|\right)\left(\sin\frac{\theta}{2}|+z\rangle - e^{i\varphi}\cos\frac{\theta}{2}|-z\rangle\right)$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2}\langle +z|+z\rangle + e^{-i\varphi}\sin\frac{\theta}{2}\cdot(-e^{i\varphi})\cos\frac{\theta}{2}\langle -z|-z\rangle =$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

$|\vec{n}\rangle$ & $|-\vec{n}\rangle$ are orthogonal



We have now made enough measurements to figure out the relationship between the probability and the angle between the analyzers. This relationship can be expressed by the function:

$$P(\theta) = \cos^2(\theta/2)$$