William & Mary

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QUANTUM MECHANICS I NOTES

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CHAPTER 2
MATRIX MECHANICS AND OPERATORS

Vectors

$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$$

$$\vec{V} = (V_x , V_y , V_z)$$

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the Unitary Vectors of the basis are

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$$\vec{V} = V_x \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\hat{x}} + V_y \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\hat{y}} + V_z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\hat{z}}$$

Column vector: matrices of 3x1 in this example

Basic operations:

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the Constant multiplies all the elements on the vector

→ Same for a Column Vector

the same if A is a matrix of $m \times n$

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & & & \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

and the product between two matrixes ; in general

if $A = m \times n$ matrix

and $B = n \times p$

Next ⟹

$A = m \times n$ matrix , $B = n \times p$ matrix , then

$$AB = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots & b_{1p} \\ b_{21} & b_{22} \cdots & b_{2p} \\ \vdots & & \\ b_{n1} & b_{n2} \cdots & b_{np} \end{pmatrix}$$

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$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{n1} \\ \\ \\ \\ \end{pmatrix}$$

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"dot product"

all the other elements

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} & \cdots \\ & & \\ & & \end{pmatrix}$$

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In that way we know how to fill all the elements

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all the other elements

☝🤓 the element of the new matrix are given by

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, p$

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Dot product is a multiplication of a row vector and column vector.

$$\vec{A} \cdot \vec{B} = (a_1, a_2, \ldots, a_n)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\vec{A} \cdot \vec{B} = \underbrace{a_1 b_1 + a_2 b_2 + \ldots + a_n b_n}_{\text{this is just a scalar Not a vector}}$$

in general

$$(x_1, x_2, x_3, \ldots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

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☝🤓 The transpose of any row vector is a column vector
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in general $[A^T]_{ij} = [A_{ji}]$

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in the same way:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}^{\dagger} = \left(x_1^*, x_2^*, x_3^*, \cdots x_n^*\right)$$

Back to Quantum Mechanics

Spin example $|\psi\rangle = C_+ |+z\rangle + C_- |-z\rangle$, we can think about the Quantum state, in a "ket" as a Column Vector in that sense:

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and for the bra representation:

$\langle +z| = (1, 0)$, $\langle -z| = (0, 1)$ and

$$\langle \psi| = C_+^* (1, 0) + C_-^* (0, 1) = (C_+^*, C_-^*)$$

• Orthonormality $\langle +z | -z \rangle = (1, 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

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- Amplitude of probability $\langle +z|\psi\rangle = (1\ ,\ 0)\begin{pmatrix}C_+\\C_-\end{pmatrix} = C_+$

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is it possible to perform the product $|+z\rangle\langle +z|$?

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$$\tilde{A}|\psi\rangle = a|\varphi\rangle$$

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In this case we said that $|\psi\rangle$ is an eigenstate of $\tilde{A}$ and a is the eigenvalue

Rotation operator

$$\hat{R}(\varphi\,\hat{i}) = e^{-i\hat{J}_x \varphi/\hbar}$$

$$\hat{R}(\theta\,\hat{j}) = e^{-i\hat{J}_y \theta/\hbar}$$

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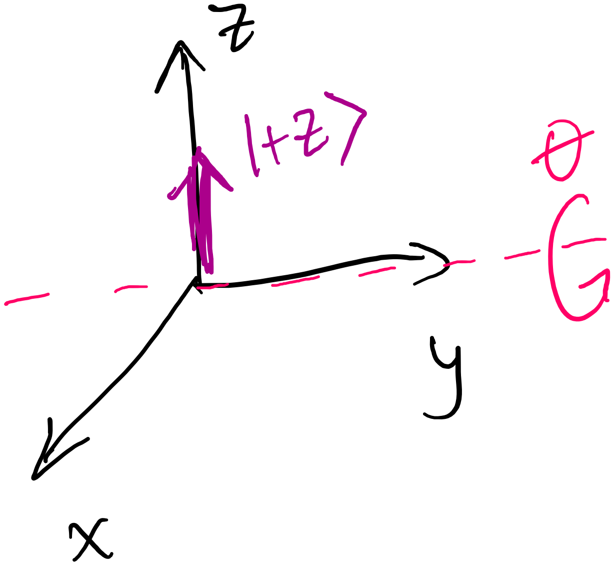
φ, θ, ϕ the angle of the rotation, $\hat{i}, \hat{j}, \hat{k}$ the axis of the Rotation and $\hat{J}_x, \hat{J}_y, \hat{J}_z$ the angular momentum operator, with eigenvalues and eigenvectors

$$\hat{J}_x |\pm x\rangle = \left(\pm\frac{\hbar}{2}\right)|\pm x\rangle \qquad \hat{J}_y |\pm y\rangle = \left(\pm\frac{\hbar}{2}\right)|\pm y\rangle$$

$$\hat{J}_z |\pm z\rangle = \left(\pm\frac{\hbar}{2}\right)|\pm z\rangle$$

Example : Rotate the $|+z\rangle$ vector about the y axis



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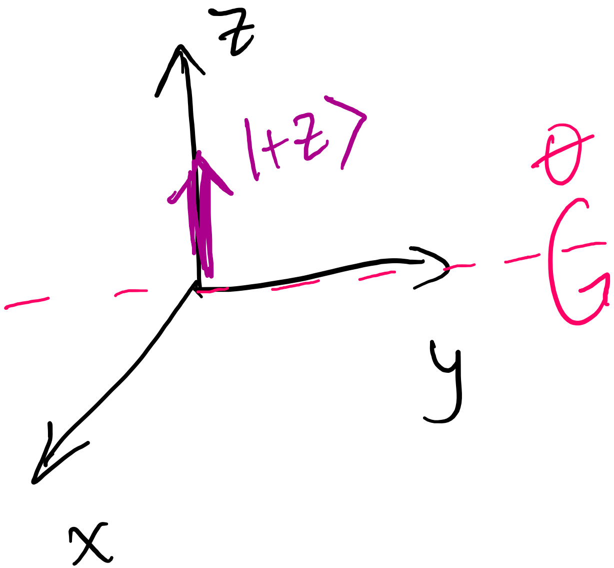
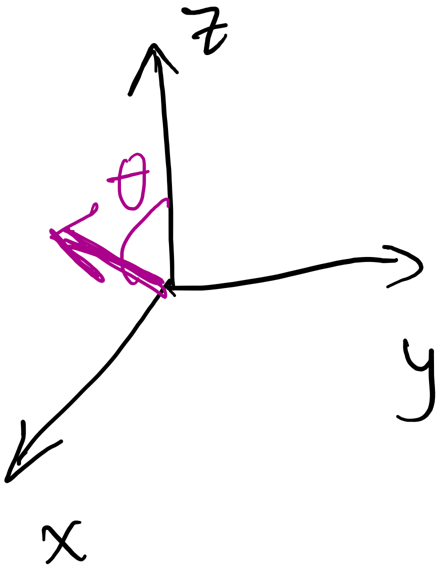
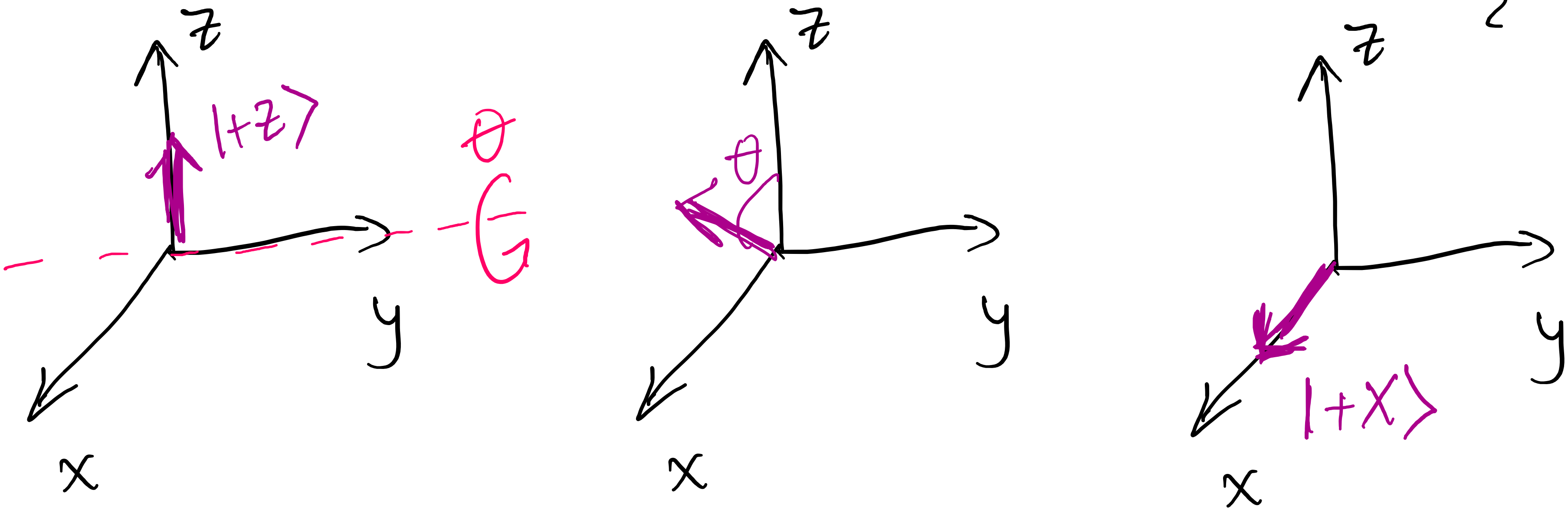




Example: Rotate the $|+z\rangle$ vector about the y axis

if $\theta = \frac{\pi}{2}$



So $|+x\rangle = \hat{R}\left(\frac{\pi}{2}\hat{j}\right)|+z\rangle$

how to operate a generall rotation

$$\hat{R}(\phi\hat{k}) = e^{-i\hat{J}_z\phi/\hbar} = \left[1 - \frac{i\phi\hat{J}_z}{\hbar} + \frac{1}{2!}\left(-\frac{i\phi\hat{J}_z}{\hbar}\right)^2 + \cdots\right]$$

how to operate a generall rotation

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it is convenient if $|\psi\rangle$ is in the z basis, because $|\pm z\rangle$ are eigenvectors of $\hat{J}_z$; so that $\hat{J}_z|\pm z\rangle = \left(\pm\frac{\hbar}{2}\right)|\pm z\rangle$

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Something to Note: $[\hat{J}_z]^N|\psi\rangle = \underbrace{\hat{J}_z\cdot\hat{J}_z\cdots\hat{J}_z}_{\text{N repetitions of the operator}}|\psi\rangle$