Problem 1(40 points)

A particle of mass m is trapped inside the infinite square well potential well of length L. You may remember that the stationary states of a particle inside the well are: $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n x}{L}\right)$.

At time $t = 0$ it is prepared in a state described by the following wave function:
$\psi(x) = \mathcal{N}\left[\frac{2\sqrt{2}}{5}\sin\left(\frac{\pi x}{L}\right) - \frac{4}{5}\sin\left(\frac{3\pi x}{L}\right) + \frac{i}{5}\sin\left(\frac{5\pi x}{L}\right)\right]$.

(a) What is the value of the normalization constant $\mathcal{N}$.
(b) Write the time evolution of this state $\psi(x,t)$.
(c) What is the average energy $\langle E\rangle$ of the particle in the state ψ?
(d) What is the probability of finding the particle in the ground state $n = 1$? In the first excited state$n = 2$? Do these probabilities depend on time?

(e) Now, the state of the particle is changed to $\tilde{\psi}(x, t = 0) = \sqrt{\frac{2}{L}}\left[\frac{3}{5}\sin\left(\frac{\pi x}{L}\right) - \frac{4}{5}\sin\left(\frac{3\pi x}{L}\right)\right]$. What is the probability of finding a particle in this state in the first quarter of the well $0 < x < L/4$ at time $t > 0$?
Hint: you can find the following expressions useful:
$\frac{2}{L}\int_0^{L/4}\sin\left(\frac{\pi a x}{L}\right)\sin\left(\frac{\pi b x}{L}\right)dx = \frac{1}{2\pi(a-b)}\sin\left(\frac{\pi(a-b)}{4}\right) - \frac{1}{2\pi(a+b)}\sin\left(\frac{\pi(a+b)}{4}\right)$ for $a \neq b$,
and $\frac{2}{L}\int_0^{L/4}\sin^2\left(\frac{\pi a x}{L}\right)dx = \frac{1}{2} - \frac{1}{4\pi a}\sin\left(\frac{\pi a}{2}\right)$.

a) $N = \sqrt{2/L}$ since $\left(\frac{2\sqrt{2}}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 + \left|\frac{i}{5}\right|^2 = \frac{8+16+1}{25} = 1$

b) $\psi(x) = \frac{2\sqrt{2}}{5}\psi_1 - \frac{4}{5}\psi_3 + \frac{i}{5}\psi_5$

$\psi(x,t) = \frac{2\sqrt{2}}{5}e^{-iE_1t/\hbar}\psi_1 - \frac{4}{5}e^{-iE_3t/\hbar}\psi_3 + \frac{i}{5}e^{-iE_5t/\hbar}\psi_5$

where $E_n = \frac{\pi^2 n^2\hbar^2}{2mL^2}$

c) $\langle E\rangle = \frac{8}{25}\cdot E_1 + \frac{16}{25}E_3 + \frac{1}{25}E_5 = \frac{8}{25}E_1 + \frac{9\cdot 16}{25}E_1 + \frac{25}{25}E_1$

$= \frac{177}{25}E_1 \qquad E_1 = \frac{\pi^2\hbar^2}{2mL^2}$

d) $P_1 = \frac{8}{25}$ $\quad P_2 = 0$, doesn't depend on time

e) $P_{0<x<L/4} = \int_0^{L/4}|\tilde{\psi}(x)|^2dx = \frac{2}{L}\int_0^{L/4}\left(\frac{3}{5}\sin\frac{\pi x}{L} - \frac{4}{5}\sin\frac{3\pi x}{L}\right)^2dx =$

$= \frac{2}{L}\left[\frac{9}{25}\int_0^{L/4}\sin^2\frac{\pi x}{L}dx + \frac{16}{25}\int_0^{L/4}\sin^2\frac{3\pi x}{L}dx - \frac{24}{25}\int_0^{L/4}\sin\frac{\pi x}{L}\sin\frac{3\pi x}{L}dx\right] =$

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$= \frac{9}{25}\left[\frac{1}{2} - \frac{1}{4\pi}\right] + \frac{16}{25}\left[\frac{1}{2} + \frac{1}{12\pi}\right] - \frac{24}{25}\left[\frac{1}{4\pi}\right] = \frac{1}{2} - \frac{83}{300\pi}$

$0 = 2$ вод

For $t>0$

$$\psi(x,t) = \sqrt{\frac{2}{L}} \left[\frac{3}{5} \sin\frac{\pi x}{L} e^{-iE_1 t/\hbar} - \frac{4}{5} \sin\frac{3\pi x}{L} e^{-iE_3 t/\hbar} \right]$$

$$|\psi(x,t)|^2 = \frac{2}{L} \left[\frac{9}{25} \sin^2\frac{\pi x}{L} + \frac{16}{25} \sin^2\frac{3\pi x}{L} - \frac{12}{25} \sin\frac{\pi x}{L} \sin\frac{3\pi x}{L} \times \right.$$

$$\times \underbrace{\left(e^{i(E_3-E_1)t/\hbar} + e^{-i(E_3-E_1)t/\hbar} \right)}_{2\cos\left(\frac{(E_3-E_1)t}{\hbar}\right) = 2\cos\left[\frac{4\pi^2 \hbar t}{mL^2}\right]}$$

$$P^{(t)}_{0<x<L/4} = \int_0^{L/4} |\psi(x,t)|^2 dx = \frac{9}{25}\left[\frac{1}{2} - \frac{1}{4\pi}\right] + \frac{16}{25}\left[\frac{1}{2} + \frac{1}{12\pi}\right] -$$

$$- \frac{24}{25}\left[\frac{1}{4\pi}\right] \cos\frac{4\pi^2 \hbar t}{mL^2} =$$

$$= \frac{1}{2} - \frac{11}{300\pi} - \frac{6}{25\pi} \cos\left[\frac{4\pi^2 \hbar t}{mL^2}\right]$$

Problem 2 (30 points)

A neutron (spin-1/2 particle) in the state $|+x\rangle$ enters the region filled with a constant magnetic field B_0 in the z direction, so that the Hamiltonian of the particle in the magnetic field is $\hat{H} = \omega_0 \hat{S}_z$.

(a) What is the average energy of the neutron? Does it depend on time?

(b) What is the average value of the operator $\hat{S}_x$ as a function of time?

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

a) $\langle +x|\hat{H}|+x\rangle = \omega_0 \langle +x|\hat{S}_z|+x\rangle = \frac{\hbar\omega_0}{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = 0$

$\langle E\rangle = 0$, doesn't depend on time

b) State evolution

$|\pm z\rangle \rightarrow |\pm z\rangle e^{\mp i\omega_0 t/2}$

$$|\alpha(t=0)\rangle = |+x\rangle = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$$

$$|\alpha(t)\rangle = \begin{pmatrix}1/\sqrt{2}\, e^{-i\omega_0 t/2}\\ 1/\sqrt{2}\, e^{i\omega_0 t/2}\end{pmatrix}$$

$$\langle S_x(t)\rangle = \langle\alpha(t)|\hat{S}_x|\alpha(t)\rangle = \frac{\hbar}{2}\begin{pmatrix}\frac{e^{i\omega_0 t/2}}{\sqrt{2}} & \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{e^{-i\omega_0 t/2}}{\sqrt{2}}\\ \frac{e^{i\omega_0 t/2}}{\sqrt{2}}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t}\right) = \frac{\hbar}{2}\cos\omega_0 t$$

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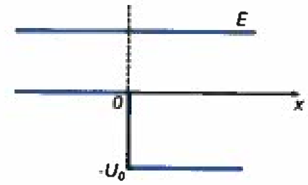
Problem 3 (30 points)

A particle with mass m and total energy $E > 0$ approaches the potential step $V(x)$ from the left, as shown:

$$V(x) = \begin{cases} 0 & x \leq 0 \\ -V_0 & x > 0 \end{cases}$$



A partial reflector is placed in the positive infinity $x \to +\infty(x)$ such that a part of the wave comes back with amplitude reflection coefficient r (in general, r is a complex number and $|r| < 1$). As a result, the wavefunction for $x > 0$ should be written in a form $Ce^{ikx} + rCe^{-ikx}$.

(a) Write down the general expression for the particle wave function ψ for all value of x.
(b) State the boundary conditions at $x = 0$.
(c) Find the probability for a particle to be reflected off the potential step at $x = 0$.
(d) Find the value(s) of E for which this reflection disappears.

a)
$$\psi(x) = \begin{cases} Ae^{ik_0x} + Be^{-ik_0x} & x<0 \\ Ce^{ikx} + rCe^{-ikx} & x>0 \end{cases} \qquad k_0 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k = \sqrt{\frac{2m(E+U_0)}{\hbar^2}}$$

b)
$$\psi(x=0-) = \psi(x=0+) \qquad A+B = C + rC$$
$$\psi'(x=0-) = \psi'(x=0+) \qquad ik_0A - ik_0B = ikC - ikrC$$

$$C = \frac{A+B}{1+r}$$

c)
$$k_0(1+r)A - k_0(1+r)B = \underbrace{k(A+B) - kr(A+B)}_{k(1-r)A + kB(1-r)}$$

$$[k_0(1+r) - k(1-r)]A = [k(1-r) + k_0(1+r)]B$$

Reflection $\quad R = \left|\frac{B}{A}\right|^2 = \left[\frac{k_0(1+r) - k(1-r)}{k_0(1+r) + k(1-r)}\right]^2$

d) $\quad R=0 \qquad k_0(1+r) = k(1-r) \qquad \frac{\cancel{2m}E}{\cancel{\hbar^2}}(1+r)^2 = \frac{\cancel{2m}(E+U_0)}{\cancel{\hbar^2}}(1-r)^2$

$$E[(1+r)^2 - (1-r)^2] = 4rE = U_0(1-r)^2$$

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Energy when no reflection $\quad \boxed{E = U_0 \frac{(1-r)^2}{4r}}$