Quantum states of light

a) Fock states — states with known number of photons $\underline{|n\rangle}$

These states are highly non-classical, i.e they don't have classical analogues

$$\hat{E}_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\hat{a}\, e^{ikz-i\omega t} + \hat{a}^+ e^{-ikz+i\omega t} \right)$$

Average value of electric field

$$\langle n|\hat{E}_x|n\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\langle n|\hat{a}|n\rangle e^{ikz-i\omega t} + \langle n|\hat{a}^+|n\rangle e^{-ikz+i\omega t} \right)$$

$$\langle n|\hat{a}|n\rangle = \sqrt{n}\, \langle n|n-1\rangle = 0$$
$$\langle n|\hat{a}^+|n\rangle = \sqrt{n+1}\, \langle n|n+1\rangle = 0$$

Average value of electric field is $\underline{\text{zero}}$!
no well-defined amplitude

Fluctuations of the electric field

$$\Delta E_x = \sqrt{\langle E_x^2\rangle - \cancel{\langle E_x\rangle}^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{2n+1}$$

Vacuum fluctuations $n=0$ $\Delta E_x^{(vac)} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$

We cannot describe a Fock state as an electric field oscillations, with a known amplitude and phase.
Energy is well-defined (i.e we can deduce the amplitude), but the phase is completely unknown.

b) Coherent state — the most laser-like state

$$\hat{E}_x \propto \hat{a}$$

So we need a state that will provides us with the most well-defined electric field value

Coherent state is an eigenstate of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

We can present $|\alpha\rangle$ as a combination of number states

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{n'=0}^{\infty} c_{n'+1}\sqrt{n'+1}\,|n'\rangle$$

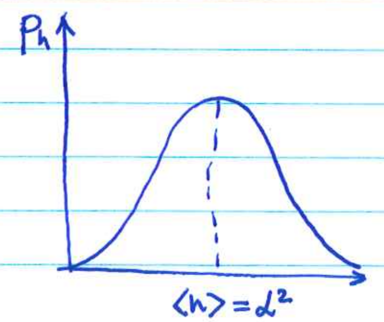
$$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

$$c_{n+1}\sqrt{n+1} = \alpha c_n \quad \Rightarrow c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n \Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

With proper normalization

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Clearly, the number of photons in a coherent state is not precisely defined



Probability to measure n photons

$$P_n = |c_n|^2 = \frac{\alpha^{2n}}{n!}$$

Average electric field

$$\langle \alpha | \hat{E}_x | \alpha \rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\underbrace{\langle \alpha | \hat{a} | \alpha \rangle}_{\alpha} e^{ikz - i\omega t} + \underbrace{\langle \alpha | \hat{a}^+ | \alpha \rangle}_{\alpha^*} e^{-ikz + i\omega t} \right)$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\alpha e^{ikz - i\omega t} + \alpha^* e^{-ikz + i\omega t} \right)$$

$$\alpha = |\alpha| e^{i\varphi}$$

$$\langle E_x \rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\alpha| \left(e^{ikz - i\omega t + i\varphi} + e^{-ikz + i\omega t - i\varphi} \right) =$$

$$= 2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\alpha| \cos(kz - \omega t + \varphi)$$

Classical expression for electric component of e-m field

Fluctuations of e-m field

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$ same as vacuum fluctuations

Coherent state is a minimum uncertainty state

Since there is always an uncertainty in measurable e-m field, amplitude and phase, all optical measurements are fundamentally limited in precision

Fundamental quantum limit is set by the uncertainty of the coherent state.