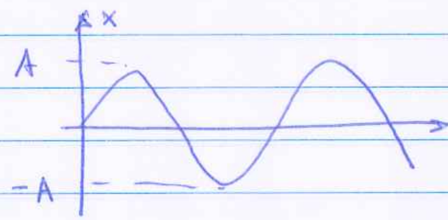
Classical harmonic oscillator

$$x(t) = A \sin \frac{2\pi}{T} t$$

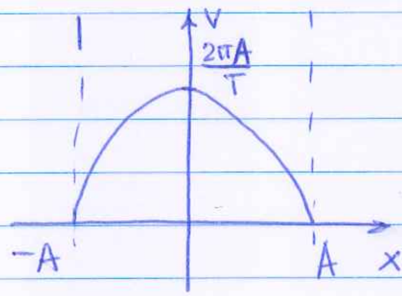


$$U(x) = \frac{1}{2} k x^2$$

$$\omega = \frac{2\pi}{T} = \sqrt{k/m}$$

$$v(t) = A \cdot \frac{2\pi}{T} \cos \frac{2\pi}{T} t = \frac{2\pi}{T} \sqrt{A^2 - \underbrace{A^2 \sin^2 \frac{2\pi}{T} t}_{x^2(t)}}$$

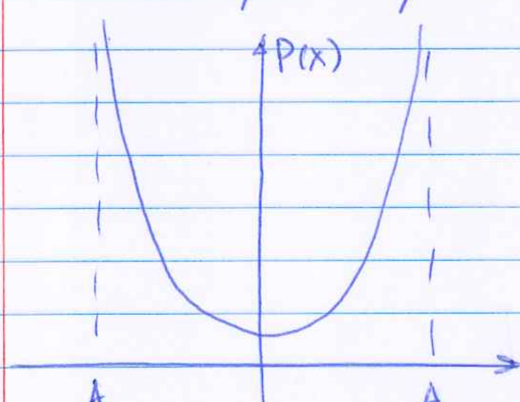
$$v(x) = \frac{2\pi}{T} \sqrt{A^2 - x^2}$$



Probability density: probability to catch an obget b/w x and $x+\Delta x$, in the limit of $\Delta x \to 0$

$$P(x, x+\Delta x) = \frac{\Delta t}{T/2} = \frac{\Delta x / v(x)}{T/2} = \frac{2 \Delta x}{\cancel{T} \cdot \frac{2\pi}{\cancel{T}} \sqrt{A^2 - x^2}} = \frac{\Delta x}{\pi \sqrt{A^2 - x^2}}$$

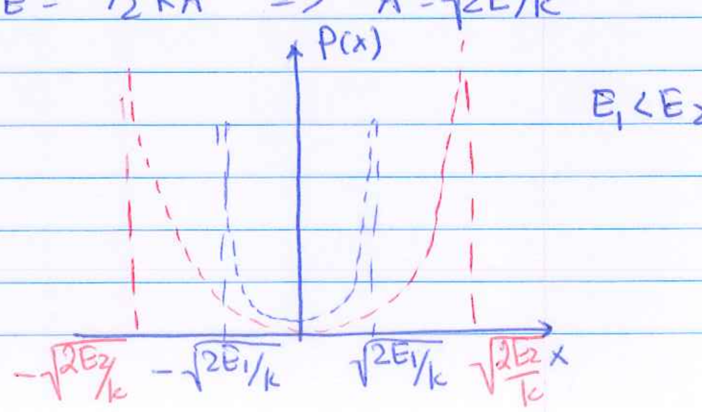
Probability density $\quad P(x) = \lim_{\Delta x \to 0} \frac{P(x, x+\Delta x)}{\Delta x} = \frac{1}{\pi \sqrt{A^2 - x^2}}$



It is easier to catch the oscillating object at the edges.

Because of the energy conservation,

$$E = \frac{1}{2} k A^2 \Rightarrow A = \sqrt{2E/k}$$



Simple harmonic oscillator (SHO)

$U(x) = \frac{1}{2}kx^2$

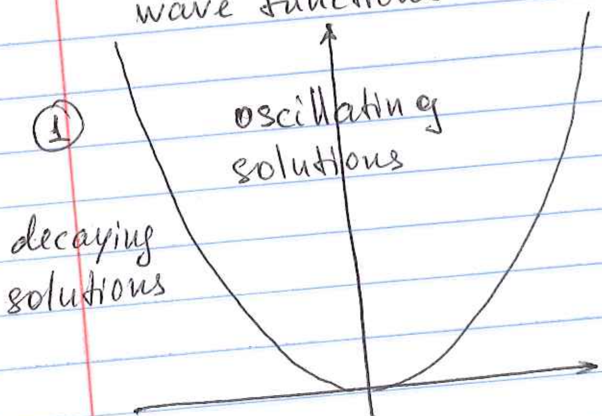
Classical frequency $\omega_0 = \sqrt{k/m}$

a mass on a string
vibrating atom in a crystal lattice
molecules

Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}kx^2\psi(x) = E\psi(x)$$

What can we guess about SHO wave functions without solving the Schrödinger eqn?

① 

② # of extrema – # of the energy level

③ Since potential is symmetric $U(x) = U(-x)$ wavefunctions are
$\psi(x) = \pm\psi(x)$
either symmetric or antisymmetric
(since $\psi''(x) = \pm\psi''(-x)$

④ Since the probability of finding a classical oscillator is higher near turning points (since it moves slower there then $|\psi(x)|^2$ should be higher near the turning points than in the cen for higher energy levels (due to the correspondence principle)

Ground state : $\underbrace{\varphi_0(x) = A e^{-\alpha x^2}}_{\text{we guessed this form}}$

$$\varphi_0'(x) = -2\alpha x \cdot A e^{-\alpha x^2} = -2\alpha x \cdot \varphi_0(x)$$

$$\varphi_0''(x) = -2\alpha \cdot A e^{-\alpha x^2} + 4\alpha^2 x^2 \cdot A e^{-\alpha x^2} = (-2\alpha + 4\alpha^2 x^2)\varphi_0(x)$$

$$-\frac{\hbar^2}{2m}(-2\alpha + 4\alpha^2 x^2)\cancel{\varphi_0(x)} + \frac{1}{2}kx^2 \cancel{\varphi_0(x)} = E_0 \cancel{\varphi_0(x)}$$

$$\underbrace{\left(\frac{\hbar^2 \alpha}{m} - E_0\right)}_{=0} + \underbrace{\left(-\frac{2\hbar^2\alpha^2}{m} + \frac{1}{2}k\right)}_{=0} x^2 = 0$$

Then $\alpha^2 = \frac{km}{4\hbar^2} = \frac{\omega_0^2 m^2}{4\hbar^2}$ $\qquad \omega_0^2 = k/m$

$$\alpha = \frac{\omega_0 m}{2\hbar} \quad \Rightarrow \quad E_0 = \frac{\hbar\omega_0}{2}$$

Ground state of any quantum oscillator has energy. Since in quantum electrodynamics electromagnetic waves are treated as oscillators, vacuum (i.e. the absence of photons) corresponds to the zeroth SHO state, and contains $\frac{1}{2}\hbar\omega_0$ of energy

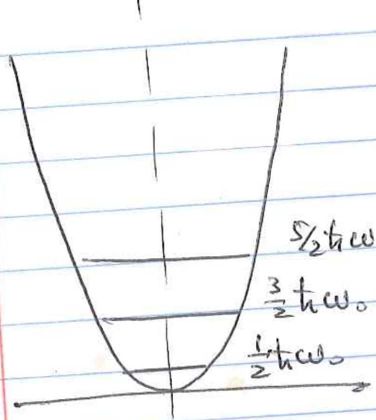
Ground state $E_0 = \frac{1}{2}\hbar\omega_0$ $\qquad \varphi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

~~Higher~~ Excited states: $E_n = (n + \frac{1}{2})\hbar\omega_0$

$$\varphi_n = \left(\frac{1}{\sqrt{\pi}\, 2^n n!}\right)^{1/2} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}} \cdot x\right)$$

$H_n(x)$ – special Hermitian polinomials.

Distinct feature of SHO – equidistant energy spectrum.



Energy difference b/w any two ~~consequ~~ consecutive states
$E_{n+1} - E_n = \hbar\omega_0$
independent on n
Looks familiar ?!

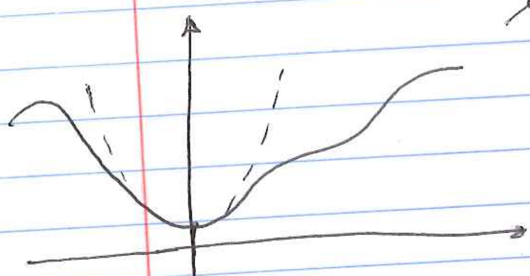
Back to black body radiation problem. Since the atoms inside crystal lattice ~~oscillate,~~ vibrate, they can be more or less accurately described by SHO picture.
Thus, if all oscillators are originally in their ground state, and the energy required to excite the oscillator to the next level, one needs to provide sufficient energy $\Delta E = \hbar\omega$

Thus, if the energy of a black body is limited, there always will be a "cut-off" for high energy radiation (small wavelength).

Simple harmonic potential is one of the most useful in physics!

Any ~~po~~ (well, almost any) potential near equilibrium can be treated as SHO!

$$U(x) = U(0) + \underbrace{\cancel{\frac{dU}{dx}(0)\cdot x}}_{=0} + \underbrace{\frac{d^2U}{dx^2}(0)\cdot x^2}_{\text{SHO}} + \left[\text{higher order terms}\right]$$



near minimum $U(x) \approx \frac{d^2U}{dx^2}(0)\, x^2$

Time dependent ~~variat~~ wave function

$$\psi_n(x,t) = \psi_n(x,t)\, e^{-E_n t/\hbar}$$

For SHO $\quad E_n = (n-1/2)\hbar\omega$

$$\psi_n(x,t) = A\cdot\psi_n(\omega)\, e^{-\hbar\omega t/\hbar} = A\psi_n(x)\, e^{-i\omega t}$$

$$\psi_n(x,t) \propto H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x\right) e^{-\frac{m\omega x}{\hbar}}\, e^{-i\omega t}$$

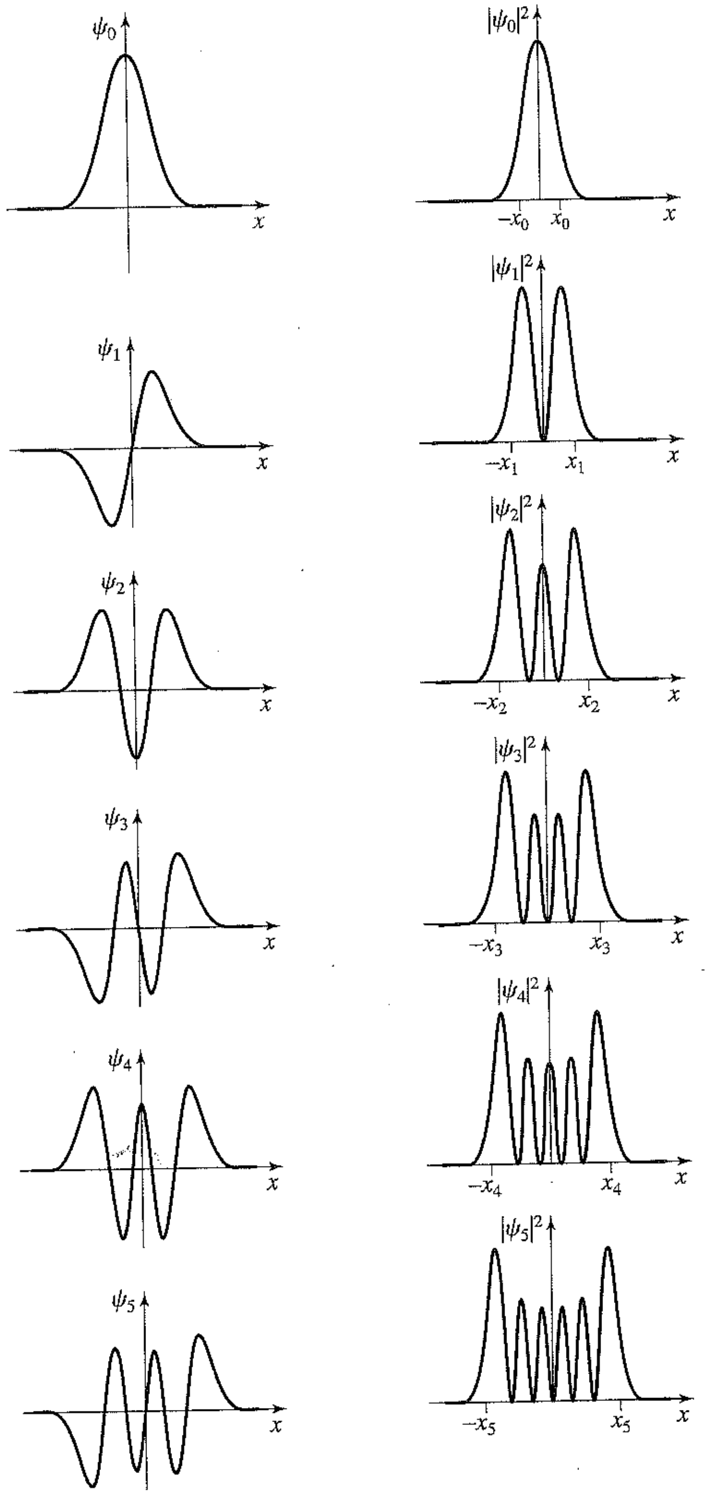


Figure 4.13 The energy eigenfunctions and probability densities for the first six energy states of the harmonic oscillator. The classical turning points x_n are determined by setting the potential energy equal to the total energy, that is $m\omega^2 x_n^2/2 = E_n$.

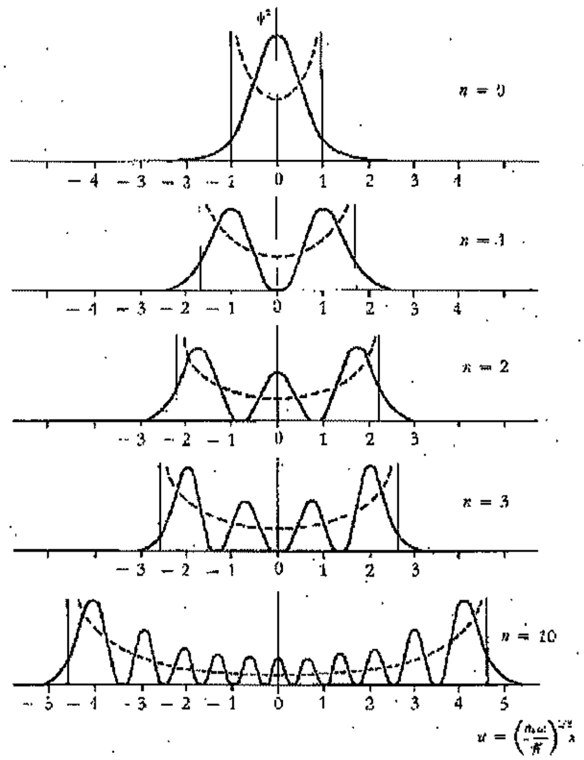


TABLE 4-1 Energy Eigenfunctions of the Simple Harmonic Oscillator

Quantum Number n	*Energy Eigenvalue* E_n	*Energy Eigenfunction* $\psi_n(x) = \left(\frac{1}{n!2^n a\sqrt{\pi}}\right)^{1/2} H_n\left(\frac{x}{a}\right) e^{-x^2/2a^2}$
0	$\frac{1}{2}\hbar\omega_o$	$\left(\frac{1}{a\sqrt{\pi}}\right)^{1/2} e^{-x^2/2a^2}$
1	$\frac{3}{2}\hbar\omega_o$	$\left(\frac{1}{2a\sqrt{\pi}}\right)^{1/2} 2\left(\frac{x}{a}\right) e^{-x^2/2a^2}$
2	$\frac{5}{2}\hbar\omega_o$	$\left(\frac{1}{8a\sqrt{\pi}}\right)^{1/2} \left[2 - 4\left(\frac{x}{a}\right)^2\right] e^{-x^2/2a^2}$
3	$\frac{7}{2}\hbar\omega_o$	$\left(\frac{1}{48a\sqrt{\pi}}\right)^{1/2} \left[12\left(\frac{x}{a}\right) - 8\left(\frac{x}{a}\right)^3\right] e^{-x^2/2a^2}$
4	$\frac{9}{2}\hbar\omega_o$	$\left(\frac{1}{384a\sqrt{\pi}}\right)^{1/2} \left[12 - 48\left(\frac{x}{a}\right)^2 + 16\left(\frac{x}{a}\right)^4\right] e^{-x^2/2a^2}$

Note: $\omega_o = (C/m)^{1/2}$; $a = (\hbar/\sqrt{mC})^{1/2} = (\hbar/m\omega_o)^{1/2}$