Maxwell's Equations

Differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \boxed{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Maxwell's Equations

Integral form

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\oint \vec{E} \cdot d\vec{l} = \boxed{-\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}}$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \boxed{\mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}}$$

The answer to ~~[illegible]~~ life, the Universe and Everything... (at least in electricity & magnetism)

Maxwell's equations!

$$\oint \vec{E}\, d\vec{A} = \frac{q}{\varepsilon_0}$$ Gauss's law

$$\oint \vec{B}\, d\vec{A} = 0$$ Gauss's law in magnetism

$$\oint \vec{E}\, d\vec{l} = -\frac{d\Phi_B}{dt}$$ Faraday's law

$$\oint \vec{B}\, d\vec{l} = \mu_0 I + \boxed{\varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}}$$ Ampere-Maxwell law

Maxwell's contribution

Differential form uses $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ charge density

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$ current density

If written as components

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \vec{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$

$$+ \vec{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

Electromagnetic wave in vacuum

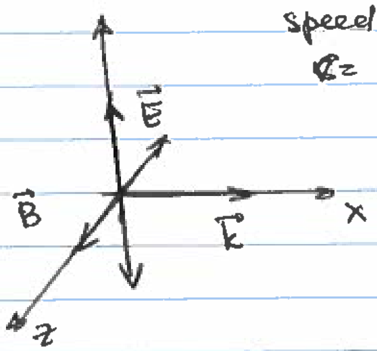
no charges, no currents

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Plane wave — describes light beam travelling along a straight line without changing

From the first two equations one can show that $\vec{E}$ and $\vec{B}$ will be perpendicular to the direction of motion (tranverse wave)

They are also perpendicular to each other



speed of light $c = \omega / k$

$$\vec{E} = \hat{j}\, E_0 \cos(kx - \omega t)$$

$k = \frac{2\pi}{\lambda}$ wave-vector, λ - wavelength

$\omega = \frac{2\pi}{T}$ frequency (in rad/s), T - period

In this geometry $\nabla \times \vec{E} = \frac{\partial E_y}{\partial x}\hat{k}$

$$\nabla \times \vec{B} = -\frac{\partial B_z}{\partial x}\hat{j}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad -\frac{\partial B_z}{\partial x} = \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t}$$

$\frac{\partial}{\partial x}$ $\qquad$ $\partial / \partial t$

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x \partial t} \qquad -\frac{\partial^2 B_z}{\partial x \partial t} = \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}$$

Wave equation $\frac{\partial^2 E_y}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}$

General form $\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t} \vec{E}$

$$\frac{\partial^2 E_y}{\partial x^2} = -E_0 k^2 \cos(kx - \omega t)$$

$$\frac{\partial^2 E_y}{\partial t^2} = -E_0 \omega^2 \cos(kx - \omega t)$$

$$\cancel{-E_0} k^2 \cancel{\cos(kx-\omega t)} = \cancel{-E_0} \omega^2 \cancel{\cos(kx-\omega t)} \cdot \varepsilon_0 \mu_0$$

$$\frac{\omega^2}{k^2} = c^2 = \frac{1}{\varepsilon_0 \mu_0} \qquad \frac{\omega}{k} = \frac{2\pi}{T} \frac{\lambda}{2\pi} = \frac{\lambda}{T}$$

$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ speed of light in vacuum

$\cancel{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$ One can construct an identical equation for $B_z = B_0 \cos(kx - \omega t)$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$-E_0 \cdot k \sin\cancel{(kx - \omega t)} = -B_0 \omega \sin\cancel{(kx - \omega t)}$$

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c$$

Electromagnetic wave carries energy!

Pointing vector $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

$|\vec{S}|$ – rate of energy flow per unit area

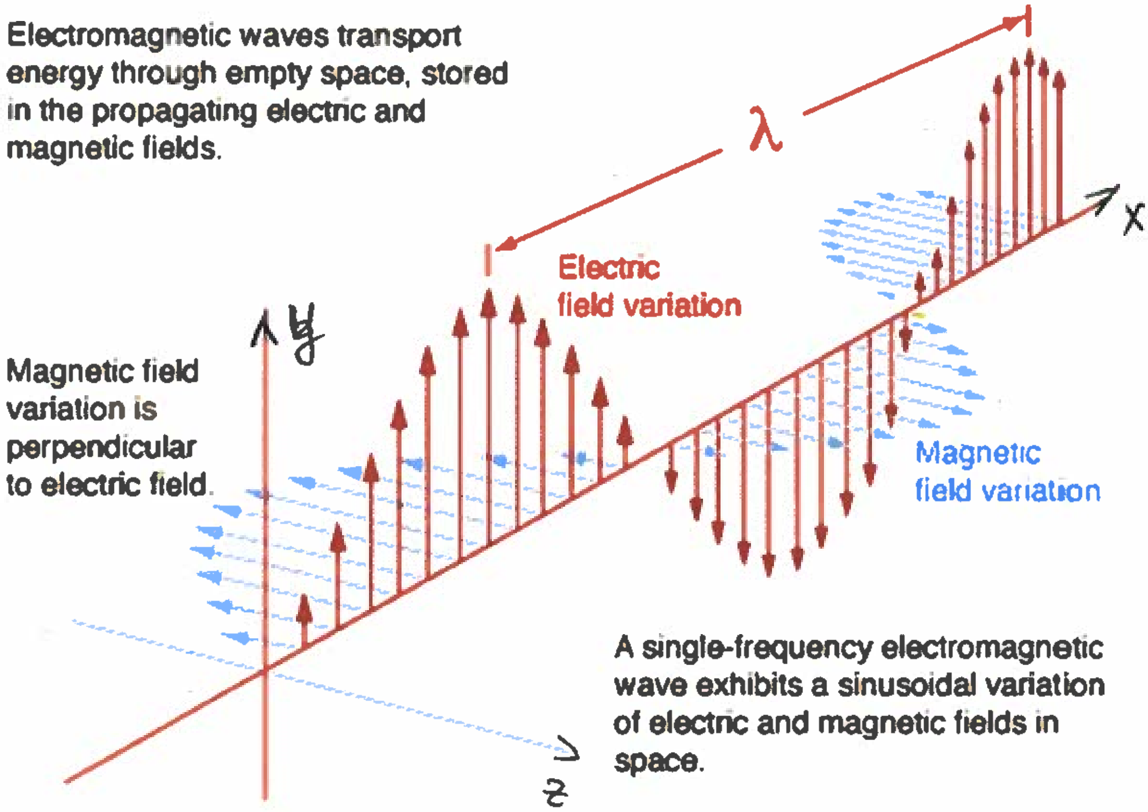
$\vec{S} \parallel \vec{k}$

$|\vec{S}| = \frac{E_0 B_0}{\mu_0} \cos^2(kx - \omega t)$ for the plane wave

Average $|\vec{S}|$ — light intensity

$$I = |\vec{S}|_{ave} = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{E_0^2}{2}$$

Electromagnetic wave



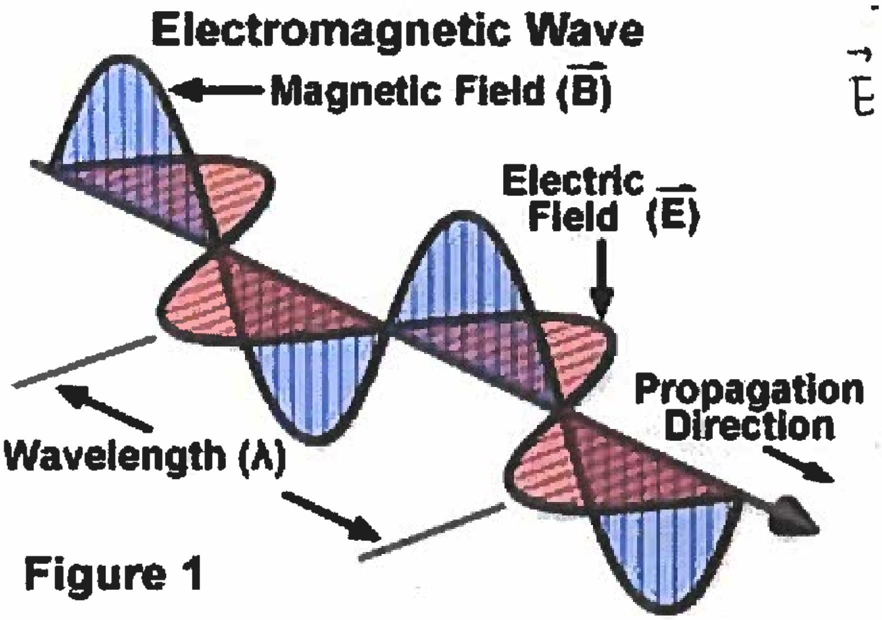


Figure 1

$$\vec{E}(x,t) = E_y(x,t)\,\hat{j}$$

$$E_y(x,t) = E_0 \cos(kz - \omega$$

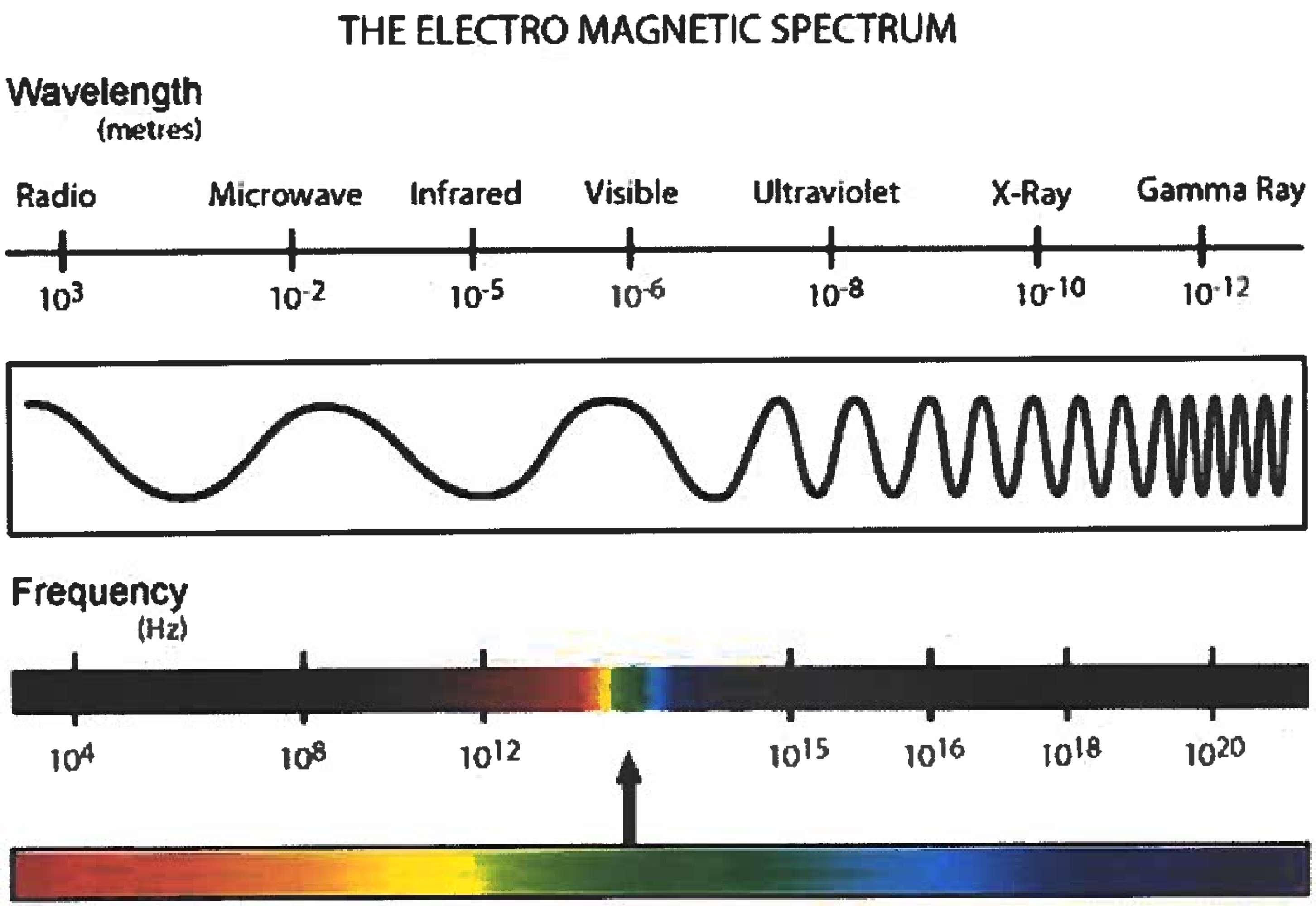
$$kz - \omega t = k\left(z - \frac{\omega}{k}t\right)$$

$$= k(z - v \cdot t)$$

velocity of the wave $\rightarrow c$

- Wave propagation direction $\vec{k}$ is along $\vec{E} \times \vec{B}$

Electromagnetic spectrum



Maxwell's equations on a plaque on his statue in Edinburgh

More details

FF-UK - Own work

Plaque showing Maxwell's equations at the Edinburgh statue.

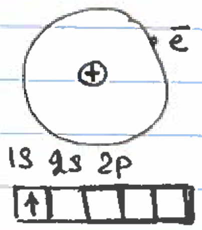
File: James Clerk Maxwell Statue Equations.jpg
Created: 2 June 2017
Uploaded: 2 June 2017

About Media Viewer

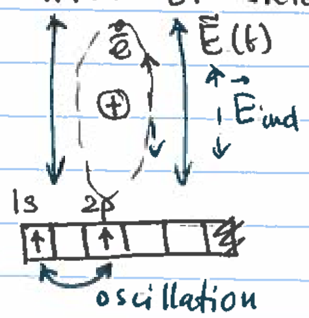
Electro magnetic waves in ~~transparent~~ materials

In majority of materials atoms / electrons interact primarily with electric component of EM field by induced dipoles

no EM field



with EM field



total fie

$\vec{E}_{tot} = \vec{E} + \vec{E}_{in}$

$E_{ind} \propto E$

includes original field and the response of the me

Maxwell's equation in the medium (without charges)

$\nabla \cdot \vec{D} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

Electric displacement vector $\vec{D} = \varepsilon_0 \varepsilon \vec{E}$

ε - dialectric constant

~~n - refractive index n =~~

Magnetizing field $\vec{H} = \frac{1}{\mu_0 \mu} \vec{B}$

μ - magnetic permeability

(for most materials $\mu \approx 1$)

Wave equation $\nabla^2 \vec{E} = \frac{1}{\varepsilon_0 \mu_0 \varepsilon \mu} \frac{\partial^2 \vec{E}}{\partial t^2}$

v^2

$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0 \varepsilon \mu}} = \frac{c}{\sqrt{\varepsilon \mu}} = \frac{c}{n}$

n - refractive index

The value of ε and n depends on material (different response of internal dipoles to the external field) and frequency of e-m wave

$n_{red} < n_{blue}$ normal dispersion (glass, water)

When light travels from one material to another, Maxwell's equations dictates how the values of different components of $\vec{E}$ and $\vec{B}$ fields are changing.

That dictates how the wave propagation direction changes

Refraction and reflection laws

Step 1

Geometrical optics → ray optics

single beams (plane waves) propagate in straight lines

Step 2

Wave optics

still plane waves, but with superposition (multiple waves add up) → interference diffraction

Step 3 (not in this course)

Quantum optics

energy carried by the wave is quantized (photons)

- quantum noise due to photon statistics

- single photon interference

- entangled photons