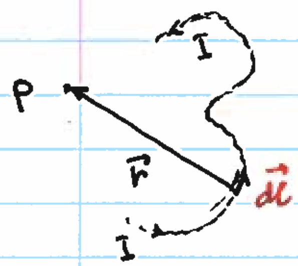
Magnetic field of electric current: common examples

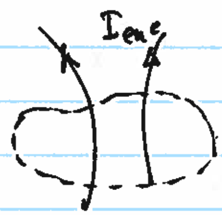
Two methods:

First $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$ or $\frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$



Total magnetic field $\vec{B} = \int d\vec{B}$ along the wire

Second: Ampere's Law: the line integral of magnetic field along a closed path is equal μ_0 times the total current enclosed by the path



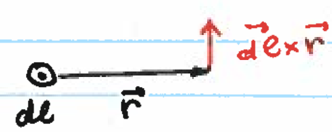
$$\oint_{\text{along the path}} \vec{B}\, d\vec{s} = \mu_0 I_{enc}$$

Useful if we can guess a good "Amperian" path with some symmetry

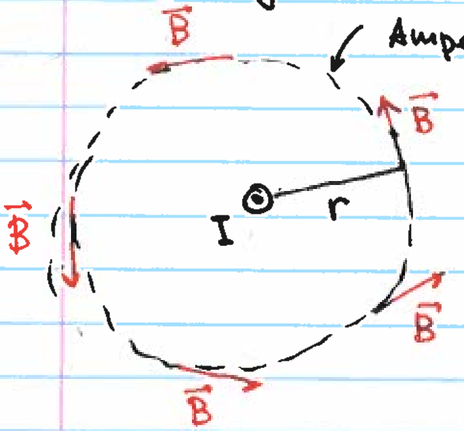
let's consider a very short element of a wire

side view

top view



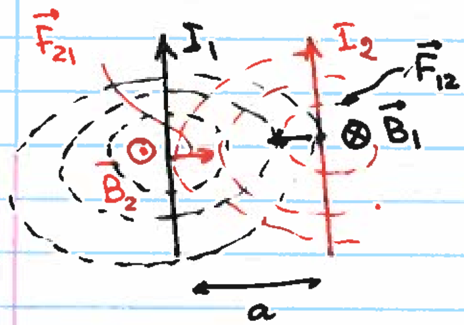
Straight very long wire



$$\oint_{circle} \vec{B}\, d\vec{s} = B \oint_{circle} d\vec{s} = B \cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Force b/w two wires



$$\boxed{F = I\, \vec{\ell} \times \vec{B}}$$

We consider I_1 creating magnetic field, and I_2 is affected by it (or vice versa)

$$B_1 = \frac{\mu_0 I_1}{2\pi a} \qquad B_2 = \frac{\mu_0 I_2}{2\pi a}$$

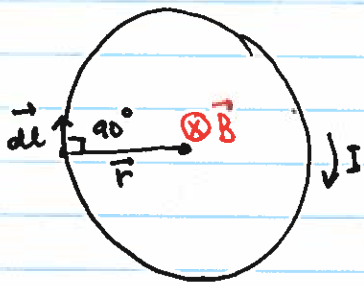
$$F_{12} = \frac{\mu_0 I_1}{2\pi a} \cdot I_2 \cdot \ell \qquad F_{21} = \frac{\mu_0 I_2}{2\pi a} I_1 \ell$$

Attractive force

per unit length $F/\ell = \dfrac{\mu_0 I_1 I_2}{2\pi a}$

Two currents flowing in the same direction attract, and two current flowing in the opposite direction repel.

A loop of wire



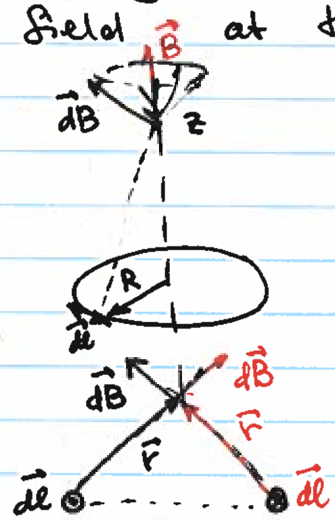
Easy to calculate the magnetic field in the center

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{|\vec{dl} \times \vec{r}|}{r^3} = \frac{\mu_0}{4\pi} I \frac{dl \cdot R}{R^3} = \frac{\mu_0}{4\pi} I \frac{dl}{R^2}$$

$$B = \oint_{\text{circumference}} \frac{\mu_0}{4\pi} I \frac{dl}{R^2} = \frac{\mu_0}{4\pi} I \frac{1}{R^2} \oint dl = \frac{\mu_0}{4\pi} \frac{I}{R^2} \cdot 2\pi$$

$$B_{center} = \frac{\mu_0}{2} \frac{I}{R}$$

Slightly more complicated calculation: magnetic field at the central axis of the loop



Individual contributions will have both components along z and perpendicular, but when added together, only the longitudinal (along z) component will survive

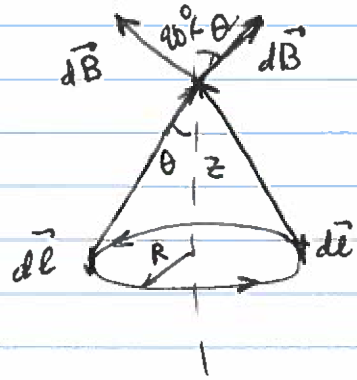
$$B_z = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}}$$ (FYI only)

If very far from a loop, we can approximate it with the field of a magnetic dipole

$\vec{\mu} = I \cdot \vec{A}$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{\mu} \cdot \hat{r})\hat{r} - \vec{\mu}}{r^3}$$ (FYI only)

What if we need to calculate the magnetic field out-of-plane ?



The contributions from current section at the opposite ends will have oppozite horizontal components that will cancel each other.

So it ~~is~~ saves us some work to find the vertical components ~~first~~ only, ~~both of them first~~, since only they will contribute

one current element $\quad dB_z = \frac{\mu_0}{4\pi} I \frac{dl}{r^2}\cos(90^\circ-\theta) \qquad r=\sqrt{z^2+R^2}$

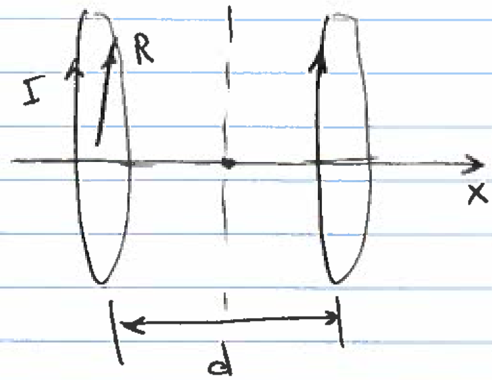
$\cos\theta = z/r \qquad \sin\theta = \frac{R}{r}$

$$B_z = \oint_{\text{circle}} \frac{\mu_0}{4\pi} I \frac{\sin\theta}{r^2}\, dl = \frac{\mu_0}{4\pi} I \frac{\sin\theta}{r^2} \underbrace{\oint dl}_{2\pi R}$$

$$B_z = \frac{\mu_0}{4\pi} I \frac{R\cdot 2\pi R}{(z^2+R^2)^{3/2}} = \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}$$

check: if $z=0$ $\quad B_z = \frac{\mu_0 I}{2R}$, as expected

Helmholtz coils



Two coils carrying identical currents (B on axis is in x-direction)

$$B = B_1 + B_2 =$$

$$= \frac{\mu_0 I R^2}{2}\left[\frac{1}{(R^2+x_1^2)^{3/2}} + \frac{1}{(R^2+x_2^2)^{3/2}}\right]$$

$$x_{1,2} = d/2 \pm \Delta x \qquad \Delta x \ll R, d$$

$$B = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(R^2+(\frac{d}{2}+\Delta x)^2)^{3/2}} + \frac{1}{(R^2+(\frac{d}{2}-\Delta x)^2)^{3/2}}\right]$$

$$\frac{1}{(R^2+(\frac{d}{2}+\Delta x)^2)^{3/2}} \approx \frac{1}{(R^2+\frac{d^2}{4}+d\Delta x)^{3/2}} = \frac{1}{(R^2+\frac{d^2}{4})^{3/2}\left(1+\frac{d}{R^2+d^2/4}\cdot\Delta x\right)^{3/2}}$$

$$\approx \frac{1}{(R^2+\frac{d^2}{4})^{3/2}}\left(1-\frac{3}{2}\frac{d}{(R^2+d^2/4)}\cdot\Delta x\right)$$

$$\frac{1}{(R^2+(\frac{d}{2}-\Delta x)^2)^{3/2}} \approx \frac{1}{(R^2+\frac{d^2}{4})^{3/2}}\left(1+\frac{3}{2}\frac{d}{(R^2+d^2/4)}\cdot\Delta x\right)$$

$$B_{tot} \approx \frac{\mu_0 I R^2}{(R^2+d^2/4)^{3/2}}$$

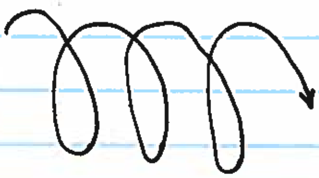
no corrections proportional to Δx

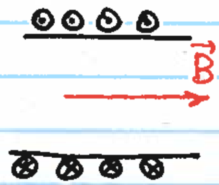
For $d=R$ no corrections $\propto (\Delta x)^2$!
Very uniform magnetic field

If two currents are counter propagating – anti-Helmholtz coils – no bias magnetic field, only gradient

$$B(\Delta x) = \frac{3\mu_0 I R^2 d}{(R^2+d^2/4)^{5/2}}\cdot\Delta x$$

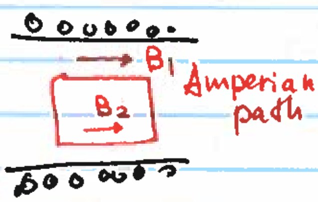
Another common way to create magnetic field is a coil → solenoid
many loops stuck together





If we assume infinite Solenoid $\vec{B}$ must be along its axis (no other directions)

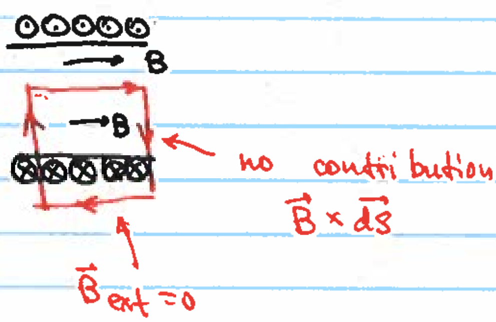
Inside the magnetic field must be uniform



$\oint \vec{B} d\vec{s} = (B_1 - B_2) \cdot \ell = 0 \Rightarrow B_1 = B_2$
no enclosed currents if the loop is fully inside.

Same logic suggests B is uniform outside as well → except if very far from the solenoid its field should drop to zero → field everywhere outside the solenoid is zero!

Solenoid creates magnetic field <u>only</u> inside



$\oint \vec{B} d\vec{s} = B \cdot \ell = \mu_0 I_{enc} = \mu_0 I \cdot N_{loops}$

$$B = \frac{\mu_0 I N_{loops}}{L}$$

$\frac{N_{loops}}{L}$ — # of loops per unit length

Capacitors create constant E-field inside and store electrostatic energy.

Solenoids (coils) are their magnetism counterparts