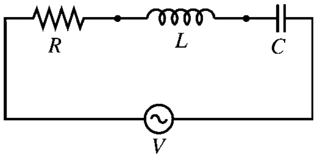
General Physics II Honors (PHYS 102H)

Problem set # 8 *(due April 18)*
All problems are mandatory, unless marked otherwise. Each problem is 10 points.
Disclamer: This assignment is a little unusual as it designed to be more of an independent research project exploring the concept of a resonance. The questions rely on knowledge obtained in class, but extend them beyond what was covered in class. Thus, it is probably the best to do the problems in order, and ask your instructor for help!

Q1: Oscillations in LC circuits. In class we briefly covered a circuit containing a capacitor and an inductor, and discussed that since such a circuit contains no losses for electromagnetic energy, the current oscillates if we start with a charged capacitor Q_0.
(a) Repeat the calculations for time-dependent current $I(t)$ and find its oscillation frequency ω_0 in terms of capacitance C and inductance L. Does this frequency depends on the initial charge?
(b) Let's imaging that you need to build a circuit that oscillates at $\omega_0 = 2\pi \cdot 10$ MHz, using commercially available components. Choose values for C and L that you can actually buy easily.

Q2 Resonant RLC circuits. We briefly discussed a general AC circuit, in which a resistor R, capacitor C and an inductor L are connected in series to an ac voltage source as shown. We will use complex exponents to describe the time-varying voltage $V(t) = V_0 e^{i\omega t}$, and to calculate the actual voltage or current we will take a real part of the complex expression.



(a) Using known expressions for the voltage drop across various circuit elements, write a relationship connecting the complex voltage and current and find the complex value of the impedance $Z(\omega)$, defined as $V(t) = Z(\omega)I(t)$.
(b) Find the absolute value of impedance $|Z(\omega)|$, and find at what value of ω it is minimum?
(c) Using the values for the capacitance and inductance you used in the first problem and $R = 1000\ \Omega$, make a graph of the $|Z(\omega)|$ dependence showing the resonance (minimum) value. You will have to decide what range of frequencies to display to clearly show the resonance. (The computer graph is preferred, but a neat hand sketch showing all important numbers is acceptable as well).

Q3: Classical analogue of light-atom interaction. (15 points) While accurate calculation of optical response of atoms to an electromagnetic field requires rather complex quantum mechanical calculation, it is possible to predict many basic properties of materials using a shockingly simple classical model, if we assume that electromagnetic field is much weaker than a Coulomb potential. In this case electron orbits are not really affected, but the electric field just make an electron "wiggle" a little by some displacement x. This small perturbation describes the induced dipole moment $\mathbf{d}(t) = -e \cdot x(t)$ responsible for optical response.

We will use the same method of presenting oscillations and waves as complex exponents, so the external electric field that an electrons "sees" is $E(t) = E_0 e^{-i\omega t}$. Since the electron is bound to its orbit, its induced motion mimics the damped mechanical oscillator:
$\frac{d^2x}{dt^2} + \frac{2}{\tau}\frac{dx}{dt} + \omega_0^2 x = -\frac{eE_0}{2m}e^{-i\omega t}$. Here τ is the lifetime of the atomic excitation, ω_0 is the natural resonance frequency (we really need Quantum to calculate those!), and m is the electron mass.
(a) Check (by substituting it into the motion equation) that the solution of this equation is $x(t) = -\frac{eE_0}{2m}e^{-i\omega t}\frac{1}{(\omega_0^2-\omega^2)-2i\omega/\tau}$
(b) We are mostly interested in frequencies relatively close to the resonance frequency ω_0, such that $|\omega_0 - \omega| \ll \omega, \omega_0$. In this approximation we can use $\omega_0 + \omega \approx 2\omega$. Show that in this approximation the induced dipole moment can be written as: $\mathbf{d}(t) = \frac{ie^2E_0\tau}{4m\omega}e^{-i\omega t}\frac{1}{1+i(\omega_0-\omega)\tau}$.

Q4: Resonant absorption and refractive index in the materials. (15 points) Here I am going to skip a few steps, but to find the optical response of the material from the Maxwell equations, we need to calculate the **electromagnetic susceptibility** $\chi(\omega)$, which is essentially the ratio between the combined induced dipole moment of N atoms per unit volume $N\mathbf{d}(t)$ and the external electric field $E(t)$:
$\chi(\omega) = \epsilon_0 N\mathbf{d}(t)/E(t) = \frac{i\epsilon_0 e^2 N\tau}{4m\omega}\frac{1}{1+i(\omega_0-\omega)\tau}$.
If we know the expression for susceptibility, we can calculate the refractive index n of the medium as:
$n(\omega) = 1 + Re(\chi(\omega))$.
We can also find the absorption coefficient α, that defines the attenuation of the electromagnetic wave intensity as it travels distance L through the material (transmission = $e^{-\alpha L}$):
$\alpha(\omega) = Im(\chi(\omega))/\lambda = Im(\chi(\omega))\frac{\omega}{2\pi c}$.
(a) Calculate the expression for the spectral absorption $\alpha(\omega)$ and find the value of the resonant absorption $\alpha_0 = \alpha(\omega_0)$ (i.e, when the e-m frequency matches the atomic resonance frequency $\omega = \omega_0$). Sketch the absorption resonance α/α_0 vs ω/ω_0 and estimate in full width half maximum width (i.e., for what $\Delta = \omega_0 - \omega$ the absorption reduces by the factor of 2 compare to its maximum resonant value.
(b) Show that the value of the refractive index $n(\omega) = 1 + \alpha_0\lambda\frac{(\omega_0-\omega)\tau}{1+(\omega_0-\omega)^2\tau^2}$ and sketch the refractive index in the same horizontal

scale as the absorption in part (a).

(c) Looking at your sketch in part (b), you may notice that absorption and dispersionseem to be related. In fact, it is possible to predict the refractive index by measuring absorption spectrum and vice versa. So if we know that the refractive index in glass is lower for red light (lower frequency) $n_{red} \approx 1.5$ than for the blue light (higher frequency) $n_{blue} \approx 1.54$, would you expect to find absorption resonance in glass in ultraviolet or infrared part of the electromagnetic spectrum?