Non-classical (squeezed state)

Number state: $|n\rangle$

known number of photons in a mode $\hat{n}|n\rangle = n|n\rangle$

Amplitude is undefined $\langle n|\hat{E}_x|n\rangle = 0$

$$\Delta E = \sqrt{\langle n|\hat{E}^2|n\rangle - \langle n|\hat{E}|n\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{2n+1}$$

Impossible to think about number state as E-M wave, as its phase is completely unknown: $\Delta n \cdot \Delta\varphi \geq 1$ $\Delta n = 0$ $\Delta\varphi = \infty$ (loose definition, as $\hat{\varphi}$ cannot be defined) Number state is an extreme example of a squeezed state, in which we have complete information about one value, while the conjugate is undefined.

Coherent state: closest analogue of a classical e-m wave: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle\alpha|E_x|\alpha\rangle = |\alpha| \underbrace{2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}}_{\text{Electric field of a single photon}} \sin(\omega t - kz - \theta)$$

$|\alpha|^2$ is average number of photons in $|\alpha\rangle$

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \quad \text{for any value of } \alpha$$

Same fluctuations as for the vacuum state $|n\rangle = 0$

$$\frac{\Delta E_x}{\overline{E}_x} \propto \frac{1}{|\alpha|}$$ The higher is photon number, the less is the effect of fluctuations

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle \qquad \hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \alpha^*\hat{a}}$$

$$E_x = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat{a}\, e^{ikz-i\omega t+\varphi} - \hat{a}^+ e^{-ikz+i\omega t+\varphi}\right) =$$

$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left[(\hat{a}-\hat{a}^+)\cos(kz-\omega t+\varphi) - i(\hat{a}+\hat{a}^+)\sin(kz-\omega t+\varphi)\right] =$$

$$= 2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left[\underbrace{\frac{1}{2}(\hat{a}+\hat{a}^+)}_{\hat{X}_1}\sin(kz-\omega t+\varphi) - \underbrace{\frac{1}{2i}(\hat{a}-\hat{a}^+)}_{\hat{X}_2}\cos(kz-\omega t+\varphi)\right]$$

$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^+)$ (intensity quadrature)

$\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^+)$ (phase quadrature)

In general: $\hat{X}_\chi = \frac{1}{2}(\hat{a}\, e^{-i\chi} + \hat{a}^+ e^{i\chi})$

$(\chi = 0 \rightarrow \hat{X}_1, \quad \chi = \pi/2 \rightarrow \hat{X}_2)$

We will see in the future that $\langle X_\chi \rangle$ and ΔX_χ can be measured experimentally

$[\hat{X}_1, \hat{X}_2] = \frac{1}{2} i$ (two orthogonal quadratures are not simultaneously measurable)

For corerent state

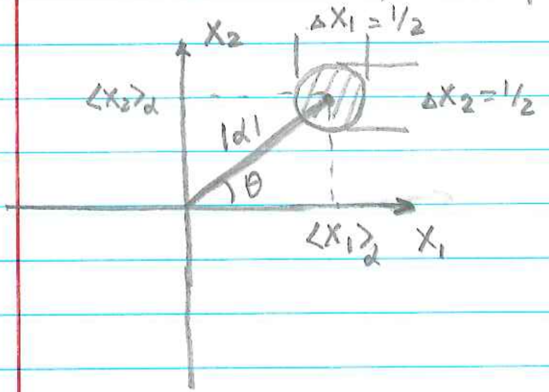
$$\langle \hat{X}_1 \rangle_\alpha = \langle \alpha | \hat{X}_1 | \alpha \rangle = \frac{1}{2}\langle \alpha | \hat{a} + \hat{a}^+ | \alpha \rangle = \frac{1}{2}(\alpha + \alpha^*) = \mathrm{Re}\,\alpha$$

$$\langle \hat{X}_2 \rangle_\alpha = \mathrm{Im}\,\alpha$$

$$\Delta X_1 = \Delta X_2 = \underset{\substack{\uparrow \\ \text{for any } \chi}}{\Delta X_\chi} = 1/2$$

Phase-space picture ("ball on a stick" picture)

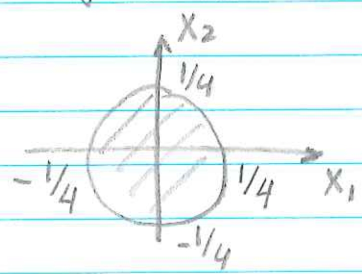
$\alpha = |\alpha| e^{i\theta}$



The "stick" represents classical (= average) instantaneous value of electric field in polar coordinates

The "ball" represents the qm fluctuations of each quadrature due to their non-communicativity.

(Coherent) vacuum state $\alpha = 0$

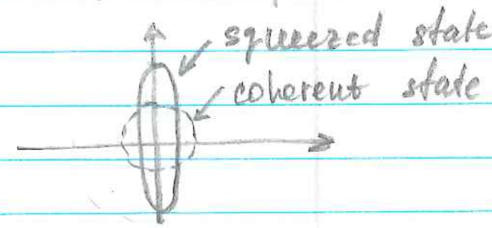


For a coherent state $\Delta X_1 = \Delta X_2 = \frac{1}{2}$
Minimum uncertainty state, that satisfies the quadrature uncertainty relationship

$$\underline{\Delta X_1 \cdot \Delta X_2 \geq \frac{1}{4}}$$

However, this restricts the product, and one quadrature may have reduced fluctuation at the expence of the other

$\Delta X_1 < 1/2$ $\Delta X_2 > 1/2$



Squeezing operator

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})} \qquad \hat{S}^+(\xi) = \hat{S}(-\xi)$$

Baker - Hausdorf lemma: $\xi = re^{i\theta}$

$$\hat{S}^+(\xi)\,\hat{a}\,\hat{S}(\xi) = \hat{a}\cosh r - \hat{a}^+ e^{i\theta}\sinh r$$

$$\hat{S}^+(\xi)\,\hat{a}^+\,\hat{S}(\xi) = \hat{a}^+\cosh r - \hat{a}\, e^{-i\theta}\sinh r$$

Squeezed vacuum state

$$|\xi\rangle = \hat{S}(\xi)\,|0\rangle$$

$$\langle\xi|\hat{a}|\xi\rangle = \langle 0|\hat{S}^+(\xi)\hat{a}\hat{S}(\xi)|0\rangle =$$

$$\langle\xi|\hat{a}^2|\xi\rangle = \langle 0|\hat{S}^+(\xi)\hat{a}\cdot\hat{a}\,\hat{S}(\xi)|0\rangle =$$

$$= \langle 0|\hat{S}^+(\xi)\,\hat{a}\,\hat{S}(\xi)\,\hat{S}^+(\xi)\,\hat{a}\,\hat{S}(\xi)|0\rangle$$

$$\langle\xi|\hat{X}_{1,2}|\xi\rangle = 0$$

$$(\Delta\hat{X}_{1,2})^2 = \frac{1}{4}\left(\cosh^2 r + \sinh^2 r \mp 2\sinh r\cosh r\cos\theta\right)$$

$$= \frac{1}{8}\left[(e^{2r} + e^{-2r}) \mp (e^{2r} - e^{-2r})\cos\theta\right]$$

For $\theta = 0$

$$(\Delta X_1)^2 = \frac{1}{4}e^{-2r}$$

$$(\Delta X_2)^2 = \frac{1}{4}e^{2r}$$

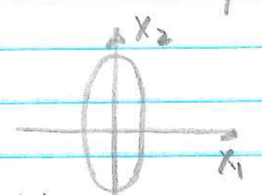
$$\Delta X_1\cdot\Delta X_2 = \frac{1}{4}$$

$\theta = \pi$

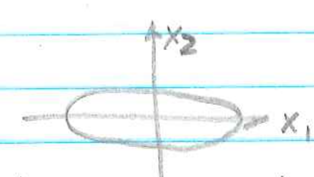
$$(\Delta X_1)^2 = \frac{1}{4}e^{2r}$$

$$(\Delta X_2)^2 = \frac{1}{4}e^{-2r}$$

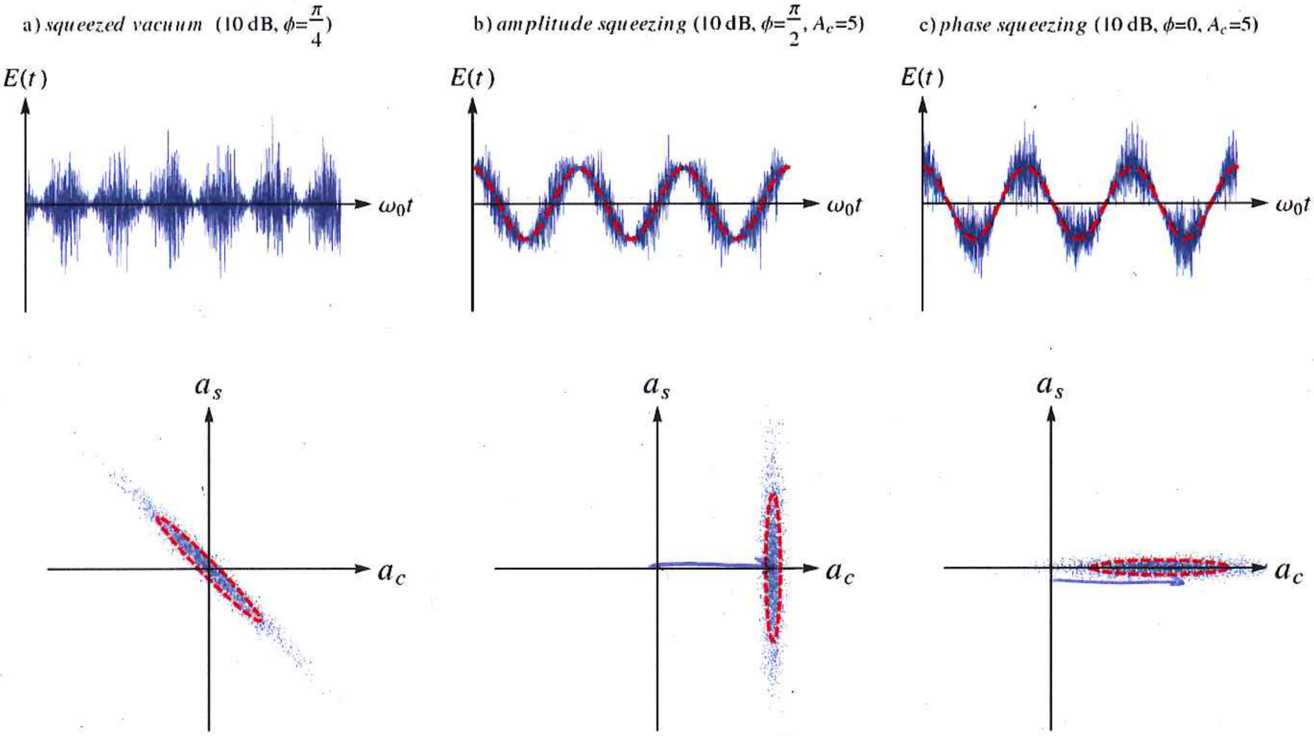
$$\Delta X_1\cdot\Delta X_2 = \frac{1}{4}$$



amplitude squeezed



phase-squeezed


a) squeezed vacuum (10 dB, $\phi=\frac{\pi}{4}$)
b) amplitude squeezing (10 dB, $\phi=\frac{\pi}{2}$, A_c=5)
c) phase squeezing (10 dB, ϕ=0, A_c=5)
$E(t)$
$\omega_0 t$
a_s
a_c

Squeezed vacuum - photon statistics

Energy of a quantized e-m field

$$\hat{H}_{EM} = \hbar\omega(\hat{n} + 1/2)$$

coherent vacuum

$$\langle E\rangle_{vacuum} = \langle 0|\hat{H}_{EM}|0\rangle = \hbar\omega\langle 0|\hat{n} + \tfrac{1}{2}|0\rangle = \underbrace{\tfrac{1}{2}\hbar\omega}_{\text{zero-point energy}}$$

for a coherent state

$$\langle E\rangle_{\alpha} = \hbar\omega\left(|\alpha|^2 + \tfrac{1}{2}\right)$$

Squeezed vacuum

$$\langle E\rangle_{SV} = \langle \xi|\hat{H}_{EM}|\xi\rangle = \hbar\omega\langle 0|\hat{S}^{+}(\xi)\,\hat{a}^{+}\hat{a}\,|\hat{S}(\xi)|0\rangle =$$

$$= \hbar\omega\langle 0|\hat{S}^{+}(\xi)\hat{a}^{+}\hat{S}(\xi)\hat{S}^{+}(\xi)\hat{a}\hat{S}(\xi)|0\rangle + \tfrac{1}{2}\hbar\omega$$

$$= \hbar\omega\langle 0|(\hat{a}^{+}\cosh r - \hat{a}e^{-i\theta}\sinh r)(\hat{a}\cosh r - \hat{a}^{+}e^{i\theta}\sinh r)|0\rangle \overset{+\frac{1}{2}\hbar\omega}{=}$$

$$= \hbar\omega\Big[\langle 0|(\hat{a}^{+})^2|0\rangle(-e^{i\theta}\cosh r\sinh r) + \langle 0|\hat{a}^2|0\rangle(-e^{-i\theta}\cosh r\sinh r)$$

$$+ \langle 0|\hat{a}^{+}\hat{a}|0\rangle\cosh^2 r + \langle 0|\underbrace{\hat{a}\hat{a}^{+}}_{\hat{a}^{+}\hat{a}+1}|0\rangle\sinh^2 r\Big] + \tfrac{1}{2}\hbar\omega$$

The only non-zero contribution!

$$\langle E\rangle_{SV} = \tfrac{1}{2}\hbar\omega + \underbrace{\sinh^2 r}_{\text{extra energy due to squeezing}}$$

Photon number distribution

$$P_n = |\langle n|\xi\rangle|^2 = |\langle n|\hat{S}(\xi)|0\rangle|^2 =$$

$$= |\langle n|\, e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}\, |0\rangle|^2 =$$

$$= |\langle n| \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})^k}{k!} |0\rangle|^2$$

Remarkably, $P_{2m+1} = 0$
only <u>even</u> number of photons can be detected!

$$P_{2m} = \binom{2m}{m} \frac{1}{\cosh r} \left(\frac{1}{2} \tanh r\right)^{2m} \qquad (\text{for } \theta = 0)$$

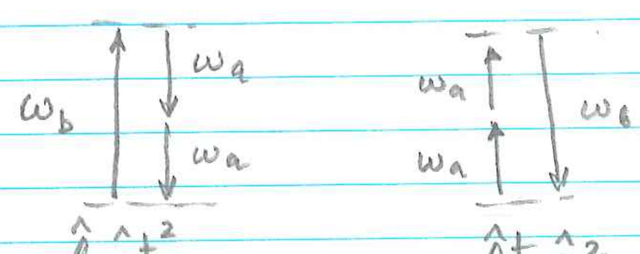
Not too surprising, considering the form of the interaction operator

$$|\psi\rangle_s = \underbrace{e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}}_{\text{result of the squeezing interaction}} |\psi\rangle$$

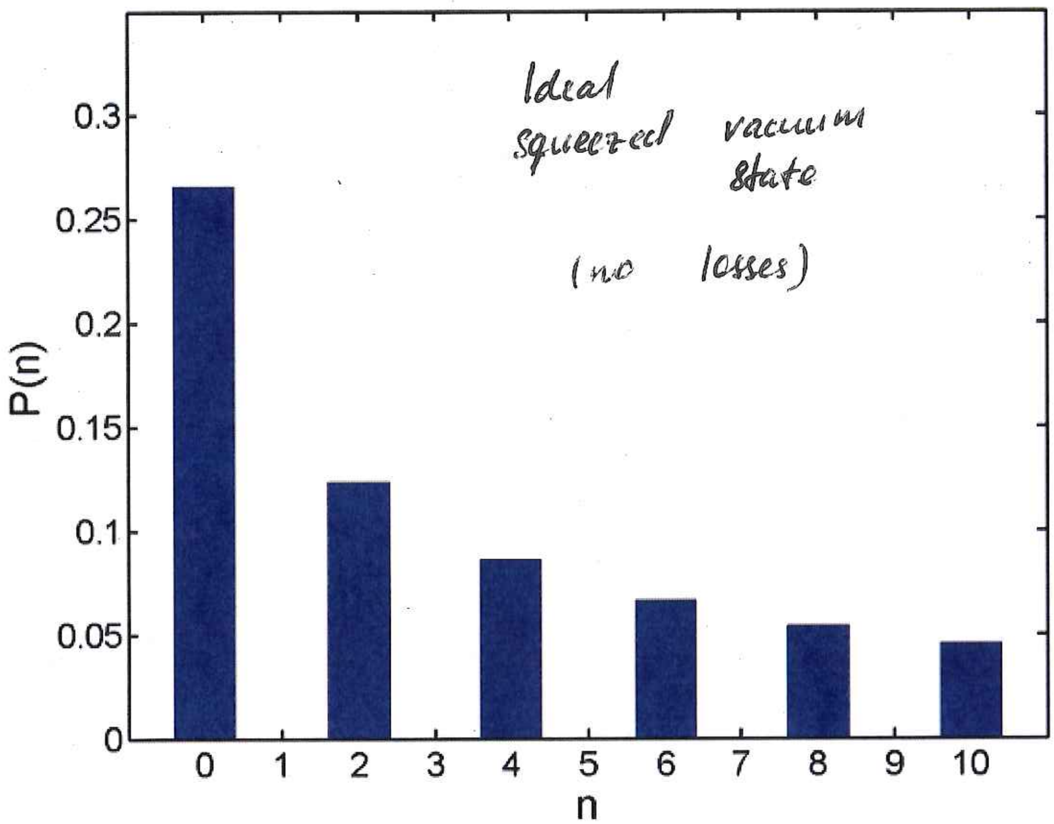
$$\hat{H}_{sq} = \chi(\beta^* \hat{a}^2 - \beta \hat{a}^{+2})$$

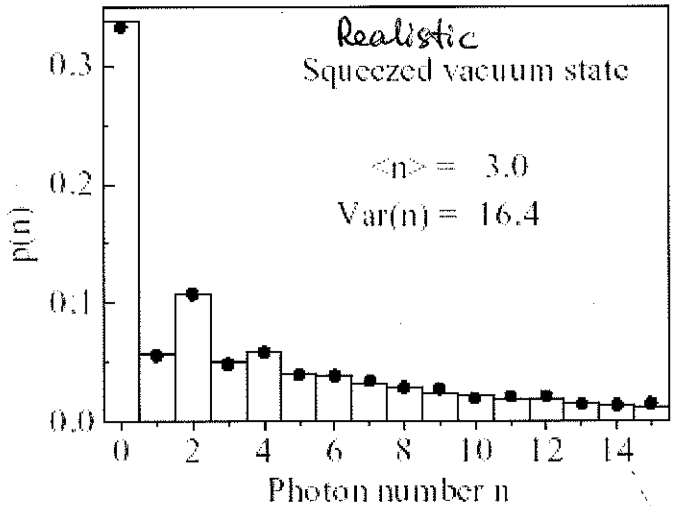
$$\Updownarrow$$

Parametric down conversion: $\hat{H}_{PDC} = \chi_{PDC}(\hat{b}^+ \hat{a}^2 - \hat{b}\hat{a}^{+2})$



$\hat{b}\,\hat{a}^{+2}$ $\qquad$ $\hat{b}^+\,\hat{a}^2$


Ideal
squeezed vacuum
state
(no losses)
0.3
0.25
0.2
0.15
0.1
0.05
0
P(n)
0 1 2 3 4 5 6 7 8 9 10
n



Any loss of photons breaks the symmetry, introducing photons into odd-number state, thus corrupting squeezing