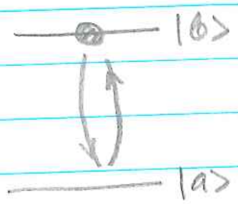
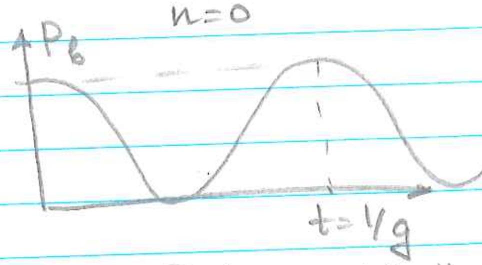
Spontaneous emission

Single-mode interaction



$$|\psi(t)\rangle = \cos(g\sqrt{n+1}\,t)|b\rangle - i\sin(g\sqrt{n+1}\,t)|a\rangle$$

$n=0$



Vacuum Rabi oscillations

This is not a common scenario, as usually a spontaneous photon is emitted and gone, never to return.

More realistic (common) scenario atom in a free space, coupled to infinite number of vacuum modes $\hat{E}_k = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, e^{i\vec{k}\vec{r}}(\hat{a}+\hat{a}^+)$

Interaction hamiltonian

$$\hat{H}_{int} = \hbar \sum_k g_k^* \hat{b}_+ \hat{a}_k e^{i(\omega_0-\omega_k)t} + g_k \hat{b}_- \hat{a}^+ e^{-i(\omega_0-\omega_k)t})$$

here g_k - coupling constant for each mode

Possible quantum state

$$|\psi(t)\rangle = c_b(t)|b,0\rangle + \sum_k c_{a,k}(t)|a,1_{\vec{k}}\rangle$$

$$c_b(t=0)=1 \qquad c_{a,k}(t=0)=0$$

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \left(\dot{c}_b |b,0\rangle + \sum_k \dot{c}_{a,k}(t) |a, 1_k\rangle\right) = \hat{H}_{int} |\psi\rangle =$$

$$= \hbar \sum_k g_k \quad c_b(t) |a, 1_k\rangle e^{-i(\omega_0 - \omega_k)t} + g_k^* c_{a,k} |b,0\rangle e^{i(\omega_0 - \omega_k)t} \Big)$$

$$\begin{cases} \dot{c}_b = -i \sum_k g_k^* e^{i(\omega_0 - \omega_k)t} \quad c_{a,k}(t) \\ \dot{c}_{a,k} = -i g_k e^{-i(\omega_0 - \omega_k)t} \quad c_b(t) \end{cases}$$

$$c_{a,k}(t) = -i g_k \int_0^t e^{-i(\omega_0 - \omega_k)t'} \quad c_b(t')\, dt'$$

$$\dot{c}_b = -\sum_k |g_k|^2 \int_0^t e^{i(\omega_0 - \omega_k)(t-t')} \quad c_b(t')dt'$$

$$\sum_k \longrightarrow 2 \frac{V}{(2\pi)^3} \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \int_0^{\pi} d\theta \sin\theta \boxed{\int_0^{\infty} k^2 dk} \qquad \frac{1}{c^3}\int_0^{\infty} \omega_k^2 d\omega_k \quad \text{as } k = \frac{\omega}{c}$$

$$g_k = \frac{1}{\hbar} \sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} \langle a| -\vec{d} \cdot \vec{e} |b\rangle =$$

$$= \sqrt{\frac{\omega}{2\varepsilon_0 \hbar V}} \underbrace{\langle a| -|d|| b\rangle}_{\wp_{ab}} \cos\theta$$

$$\int_0^{\pi} \cos^2\theta \cdot \sin\theta \, d\theta = \frac{2}{3}$$

$$\dot{c}_b = - \cancel{2} \frac{\cancel{V}}{(2\pi)^2} \frac{1}{\cancel{2}\varepsilon_0 \hbar \cancel{V}} \wp_{ab}^2 \cdot \frac{2}{3} \cdot \frac{1}{c^3} \times$$

$$\times \int_0^{\infty} \omega^3 d\omega \int_0^t e^{i(\omega_0 - \omega)(t-t')} \quad c_b(t')\, dt'$$

$$\int_0^{\infty} \omega^3 e^{i(\omega_0 - \omega)(t-t')} d\omega \approx \omega_0^3 \int_{0 \to -\infty}^{\infty} e^{i(\omega_0 - \omega)(t-t')} d\omega$$

$$\approx \omega_0^3 \cdot 2\pi \, \delta(t-t')$$

$$\dot{c}_b = -\frac{1}{4\pi\epsilon_0}\frac{4\omega_0^3\, p_{ab}^2}{3\hbar c^3}\, c_b(t) = -\frac{\Gamma}{2} c_b(t)$$

$$\Gamma = \frac{1}{4\pi\epsilon_0}\frac{4\omega_0^3\, p_{ab}^2}{3\hbar c^3}$$ the rate of spontaneous emission

$$c_b(t) = e^{-\Gamma/2\, t} \qquad P_b(t) = e^{-\Gamma t}$$

the population of the excited state decays with lifetime $\tau = 1/\Gamma$

Can we say anything about emitted light?

$$c_{a,k} = -i g_k \int_0^t dt'\, e^{-i(\omega_0-\omega_k)t' - \Gamma t'/2}$$

$$= g_k \left[\frac{1 - e^{-i(\omega_0-\omega_k)t - \Gamma t/2}}{(\omega_k-\omega_0) + i\Gamma/2}\right]$$

$$|\psi(t)\rangle = e^{-\Gamma t/2}|b,0\rangle + |a\rangle \sum_k g_k \left[\frac{1 - e^{-i(\omega_0-\omega_k)t - \Gamma t/2}}{(\omega_k-\omega_0) + i\Gamma/2}\right]|1_k\rangle$$

For $t \gg \tau = 1/\Gamma$ we can be fairly sure the atom is in the ground state

$$|\psi(t\gg\tau)\rangle \simeq |a\rangle \sum_k g_k \frac{1}{(\omega_k-\omega_0)+i\Gamma/2}|1_k\rangle$$

one photon is emitted

$$|\gamma_{sp}\rangle = \sum_k g_k \frac{1}{(\omega_k-\omega_0)+i\Gamma/2}|1_k\rangle$$

How much light is emitted in each mode?

$$\langle \gamma_{sp} | \hat{E}^{(-)}_{k_0} \hat{E}^{(+)}_{k_0} | \gamma_{sp} \rangle \propto \langle \gamma_{sp} | \hat{n}_{k_0} | \gamma_{sp} \rangle =$$

$$= |g_{k_0}|^2 \frac{1}{(\omega_{k_0} - \omega_0)^2 + \Gamma^2/4}$$

Spectrum of spontaneous emission