Coherent states

$$E_x = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat{a}\,e^{ikz-i\omega t} - \hat{a}^+ e^{-ikz+i\omega t}\right)$$

The number states we've discussed are highly non-classical, i.e. they do not have classical analogues

Measured mean amplitude of e-m field

$$\langle E_x\rangle = \langle n|E_x|n\rangle = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\overbrace{\langle n|a|n\rangle}^{=0} e^{ikz-i\omega t} - \overbrace{\langle n|a^+|n\rangle}^{=0} e^{-ikz+i\omega t}\right)$$

$= 0$ no well-defined amplitude

Fluctuations of electro-magnetic field

$$\Delta E_x = \sqrt{\langle E_x^2\rangle - |\langle E_x\rangle|^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\langle n|\,|\hat{a}e^{ikz-i\omega t} - \hat{a}^+ e^{-ikz+i\omega t}|^2|n\rangle}$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{\langle n|\,\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}|n\rangle} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{2n+1}$$

Note that even for $|0\rangle$ vacuum state

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} > 0$$

Vacuum fluctuations (now vacuum is not nothingness, it is alive and wiggles)

So what state would be the closest analogue of the classical e-m wave?

Coherent states are the eigenstates of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

If $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \Rightarrow \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle$

$$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$$

$$c_{n+1}\sqrt{n+1} = \alpha c_n$$

$$\Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \langle\alpha|\alpha\rangle = 1 = |c_0|^2 \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = e^{|\alpha|^2}$$

since $\langle n|n'\rangle = \delta_{nn'}$

$$|c_0|^2 \cdot e^{|\alpha|^2} = 1 \Rightarrow |c_0| = e^{-|\alpha|^2/2} = c_0$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$$

Average value of the electric field

$$\langle\alpha|E_x|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\underbrace{\langle\alpha|\hat{a}|\alpha\rangle}_{\alpha|\alpha\rangle} e^{ikz-i\omega t} - \underbrace{\langle\alpha|\hat{a}^+|\alpha\rangle}_{\alpha^*\langle\alpha|} e^{-ikz+i\omega t}\right)$$

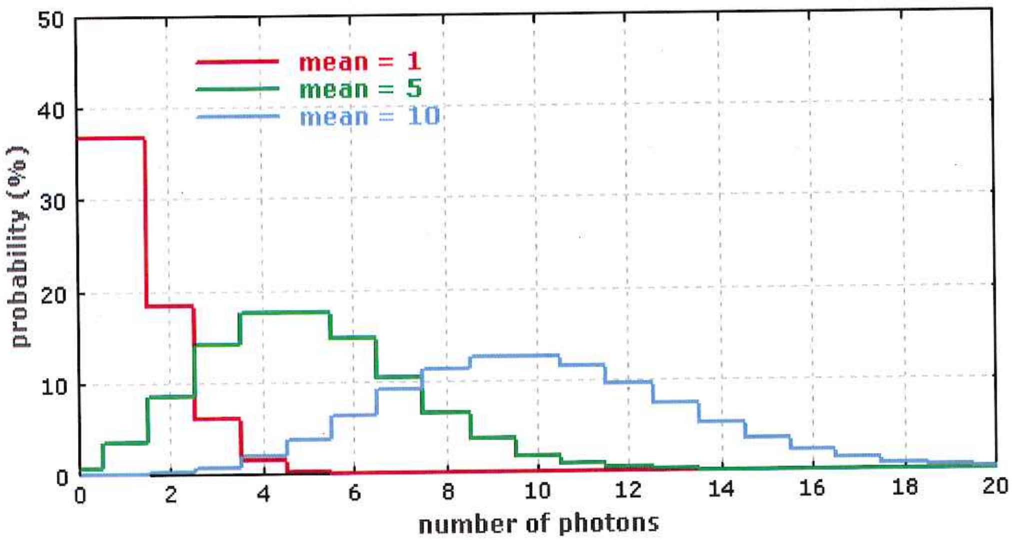
$$= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\alpha e^{ikz-i\omega t} - \alpha^* e^{-ikz+i\omega t}\right) =$$

$$\alpha = |\alpha| e^{i\varphi} \qquad \alpha^* = |\alpha| e^{-i\varphi}$$

$$= 2|\alpha|\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sin(kz - \omega t + \varphi)$$

Total energy $\frac{1}{2}\int \varepsilon_0 |E|^2 dV = \underbrace{|\alpha|^2}_{\text{"average" number of photons}} \hbar\omega$

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2$$



Photon distribution in coherent states with different mean value of photon $|\alpha|$

Electric field fluctuations

$$\langle\alpha|E^2|\alpha\rangle = -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\langle\alpha|\,\hat{a}^2 e^{2(ikz-i\omega t)} + \hat{a}^{+2} e^{-2(ikz-i\omega t)} - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}\,|\alpha\rangle$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(\alpha^2 e^{2(ikz-i\omega t)} + \alpha^{*2} e^{-2(ikz-i\omega t)} - 1 - 2|\alpha|^2\right) =$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(\underbrace{2|\alpha|^2\cos 2(kz-\omega t+\varphi) - 2|\alpha|^2}_{-4|\alpha|^2\sin^2(kz-\omega t+\varphi)} - 1\right)$$

$$\langle\alpha|E^2|\alpha\rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(1 + 4|\alpha|^2\sin^2(kz-\omega t+\varphi)\right)$$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

same fluctuation as in vacuum state

Coherent state is a minimum uncertainty state.

Coherent state is a displaced vacuum state

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$$

Displacement operator

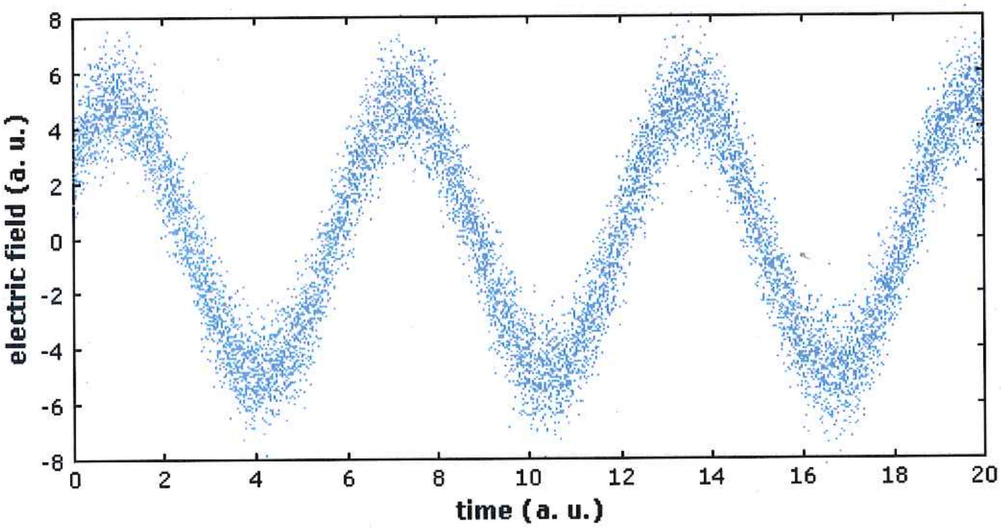
$$\hat{D}(\alpha) = e^{\alpha\hat{a} - \alpha^*\hat{a}^+}$$

$\hat{D}$ is a unitary operator

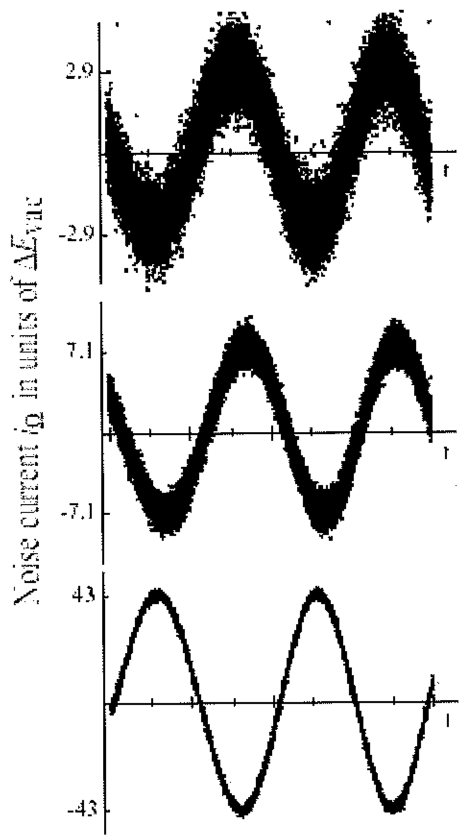
$$\hat{D}(\alpha)\,\hat{D}^+(\alpha) = \hat{D}^+(\alpha)\,\hat{D}(\alpha) = 1$$

since

$$\hat{D}^+(\alpha) = \left(e^{\alpha\hat{a} - \alpha^*\hat{a}^+}\right)^+ = e^{\alpha^*\hat{a}^+ - \alpha\hat{a}} = \hat{D}(-\alpha)$$



Coherent state = "fuzzy" electromagnetic wave



Increasing strength of e-m field

Since the uncertainty stays the same as amplitude grows, its effect becomes less and less noticable.

Vacuum fluctuations

For a single mode $E_n = \hbar\omega(n+\frac{1}{2})$
For a vacuum state $n=0$
Zero point energy
$$E_{ZPE} = \sum_{mode} \frac{1}{2}\hbar\omega \longrightarrow \infty$$

(Too) Simple solution → renormalization
(just shift the level from which we count energy)
The effect of fluctuations is directly observable

a) Spontaneous emission : electron in excited states interact with vacuum fluctuation and, as a result, change their energy level, emitting thermal radiation

b) Since there is always uncertainty in measurable e-m field amplitude, all optical measurements are fundamentally limited in precision

c) Lamb shift
Experimentally $2S_{1/2}$ and $2P_{1/2}$ states in H atom are split by ~ 1 GHz
In a semiclassical approximation, they must be degenerate

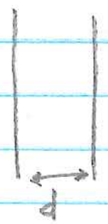
Vacuum fluctuations make an electron to randomly fluctuate from its equilibrium position, changing its energy in the Coulumb potential

$$\Delta E = \frac{1}{6} \langle (\Delta r)^2 \rangle \cdot 4\pi e^2 \underbrace{|\psi_{nml}(r=0)|^2}_{\substack{=0 \text{ for all states except} \\ \text{for } l=0 \text{ (S state)}}}$$

d) Casimir force

$$E_{ZPE} = \sum_{\text{modes}} \frac{1}{2} \hbar \omega_i$$

Two perfectly conducting parallel plates



if $d \sim \lambda$, only the wavevector

$k_z = \frac{\pi n}{d}$ are possible

$$\omega_n = c|k|_n = c\sqrt{k_x^2 + k_y^2 + (\pi n/d)^2}$$

$$E_{ZPE}^{(in)} = \int dk_x dk_y \sum_n \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + \left(\frac{\pi n}{d}\right)^2}$$

outside $\rightarrow$ no restrictions

$$E_{ZPE}^{(out)} = \int dk_x dk_y dk_z \left(\frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2}\right)$$

$$U = E_{ZPE}^{(in)} - E_{ZPE}^{(out)} = \{\text{after tedious calculations}\}$$

$$= \frac{\pi^2 \hbar c}{720 d^3} L^2$$

Casimir force (per unit area)

$$F = \frac{1}{L^2}\frac{dU}{dd} = -\frac{\pi^2 \hbar c}{240 d^4}$$