Zee 1.5–1.7
Weinberg, 4.1–4.4

Scalars, Vectors and Tensors under General Coord. Transfs.

Contravariant Vectors transform like dx^μ:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

Covariant Vectors transform like:

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu$$

Tensors transform with additional factors of $\frac{\partial x}{\partial x'}$ or $\frac{\partial x'}{\partial x}$:

$$T'^\mu{}_\nu{}^\lambda = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\lambda}{\partial x^\gamma} T^\alpha{}_\beta{}^\gamma$$

Scalars are invariant: $\phi' = \phi$

Metric: $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$

Inverse metric: $g^{\lambda\mu}$, $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda{}_\nu$

If $g'^{\lambda\mu} = \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\beta} g^{\alpha\beta}$ (2,0) tensor, then

$$g'^{\lambda\mu} g'_{\mu\nu} = \left(\frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\beta} g^{\alpha\beta}\right)\left(\frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\delta}{\partial x'^\nu} g_{\gamma\delta}\right)$$

$$= \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\delta}{\partial x'^\nu} \underbrace{\delta^\gamma_\beta\, g^{\alpha\beta} g_{\gamma\delta}}_{g^{\alpha\gamma} g_{\gamma\delta} = \delta^\alpha_\delta} \qquad \text{using } \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x'^\mu} = \delta^\gamma_\beta$$

$$= \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\lambda{}_\nu$$

$\Rightarrow g'^{\lambda\mu} g'_{\mu\nu} = \delta^\lambda{}_\nu$, $g'^{\lambda\mu}$ is the inverse of $g'_{\mu\nu}$.

Tensors can be added, multiplied, and indices lowered and raised with the metric or inverse metric (respectively), yielding new tensors.

Examples:

$C^\mu = A^\mu + B^\mu$ vector

$D^{\mu\nu} = A^\mu B^\nu$ tensor

$E^\mu = g^{\mu\nu} E_\nu$ contravariant vector

$E_\mu = g_{\mu\nu} E^\nu$ covariant vector

$F = \underbrace{A^\mu B_\mu}_{\text{contracted indices}} \equiv A^\mu B^\nu g_{\mu\nu} = A_\mu B_\nu g^{\mu\nu}$ scalar

<u>Exercise</u>: Show that the tensors in these examples transform as claimed.

Note that indices are now contracted with $g_{\mu\nu}$, not $\eta_{\mu\nu}$.

Weinberg 4.4-4.8, Zee I.5

Tensor Densities - transform with extra factors of $\det\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$ or $\det\left(\frac{\partial x'^\rho}{\partial x^\mu}\right)$ under coordinate transformations $x \to x'$.

Example: $g \equiv |\det \underbrace{g_{\mu\nu}}_{\text{metric}}|$ (The notation is different than for any other tensor. $T \equiv T^\mu{}_\mu$ is a trace. For the metric $g^\mu{}_\mu = 4$.)

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

$$\det g'_{\mu\nu} = \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^2 \det g_{\rho\sigma}$$

$$= \frac{1}{\det\left(\frac{\partial x'}{\partial x}\right)^2} \det(g_{\rho\sigma}) \to \text{tensor density of weight } -2$$

$\det\left(\frac{\partial x'}{\partial x}\right)$ is the Jacobian of the transformation $x \to x'$

The usual volume measure d^4x is a scalar density of weight 1:

$$d^4x' = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| d^4x$$

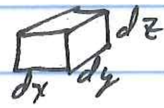
The measure $\boxed{\sqrt{g}\, d^4x}$ is a scalar. Integrals of scalars against this measure will be coordinate invariant. (This also gives the volume element in an arbitrary coordinate system.)

Example: $\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$

Levi-Civita tensor (density)

tensor density of weight -1.

Example: In 3D Euclidean space, the volume element in Cartesian coordinates is $\boxed{d^3x' = dx\,dy\,dz}$



In spherical coordinates, $x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

Define $x^1 = r,\ x^2 = \theta,\ x^3 = \varphi$
$x'^1 = x,\ x'^2 = y,\ x'^3 = z$

$$\frac{\partial x'}{\partial x} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$\det\left(\frac{\partial x'}{\partial x}\right) = r^2\sin\theta \qquad \text{(Exercise)}$$

$$\Rightarrow \boxed{d^3x' = r^2\sin\theta\, dr\, d\theta\, d\varphi}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \rightarrow g'_{ij} = \delta_{ij}, \quad \det(g') = 1$$

$$\boxed{\sqrt{g'}\, d^3x' = dx\,dy\,dz}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2 \Rightarrow g_{ij} \sim \begin{pmatrix} 1 & & 0 \\ & r^2 & \\ 0 & & r^2\sin^2\theta \end{pmatrix}$$

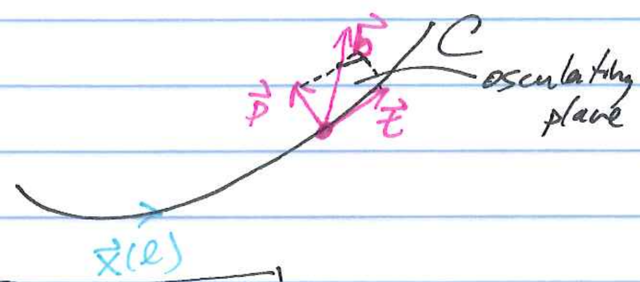
$$\det(g) = r^4\sin^2\theta, \quad \sqrt{g} = r^2\sin\theta$$

$$\boxed{\sqrt{g}\, d^3x = r^2\sin\theta\, dr\, d\theta\, d\varphi}$$

Zee 1.7

Classical Differential Geometry

Consider a curve C given by $\vec{x}(l)$ parametrized by the arc length l along the curve.



$\boxed{\vec{t} \equiv \dot{\vec{x}} = \frac{d\vec{x}}{dl}}$ = unit tangent vector

$$dl^2 = d\vec{x}\cdot d\vec{x} \Rightarrow \vec{t}\cdot\vec{t} = 1 \quad \text{(unit vector)}$$

$$\frac{d}{dl}(\vec{t}\cdot\vec{t}) = 2\,\vec{t}\cdot\dot{\vec{t}} = 0$$

Unit vector $\vec{p}(l)$ such that $\boxed{\dot{\vec{t}} = \kappa\vec{p}}$

$\vec{p} \equiv$ principal normal vector, $\boxed{\vec{p}\cdot\vec{t} = 0}$.

$\boxed{\kappa = |\dot{\vec{t}}|}$.

$\boxed{\vec{b}(l) \equiv \vec{t}(l)\times\vec{p}(l)}$ binormal vector, $\boxed{\vec{b}\cdot\vec{b} = 1}$

$$\vec{t}\cdot\vec{b} = \vec{t}\cdot(\vec{t}\times\vec{p}) = 0 \Rightarrow \frac{d}{dl}(\vec{t}\cdot\vec{b}) = \dot{\vec{t}}\cdot\vec{b} + \vec{t}\cdot\dot{\vec{b}} = 0$$

$\dot{\vec{t}}\cdot(\vec{t}\times\vec{p}) + \vec{t}\cdot\dot{\vec{b}} = 0$. Using $\dot{\vec{t}} = \kappa\vec{p}$, $\dot{\vec{t}}\cdot(\vec{t}\times\vec{p}) = 0$

$$\left.\begin{array}{l} \Rightarrow \boxed{\vec{t}\cdot\dot{\vec{b}} = 0} \\ \frac{d}{dl}(\vec{b}\cdot\vec{b}) = 2\,\vec{b}\cdot\dot{\vec{b}} = 0 \Rightarrow \boxed{\vec{b}\cdot\dot{\vec{b}} = 0} \end{array}\right\} \boxed{\dot{\vec{b}} = -\tau\vec{p}}$$

$$|\tau| = |\dot{\vec{b}}|$$

$$\vec{p} = \vec{b} \times \vec{t}$$

$$\dot{\vec{p}} = \dot{\vec{b}} \times \vec{t} + \vec{b} \times \dot{\vec{t}} = -\tau \vec{p} \times \vec{t} + \vec{b} \times \kappa \vec{p}$$

$$\boxed{\dot{\vec{p}} = \tau \vec{b} - \kappa \vec{t}}$$

Hence, we have found

$$\left.\begin{aligned} \dot{\vec{t}} &= \kappa \vec{p} \\ \dot{\vec{b}} &= -\tau \vec{p} \\ \dot{\vec{p}} &= \tau \vec{b} - \kappa \vec{t} \end{aligned}\right\} \text{ Frenet-Serret equations.}$$

Define $\Psi \equiv \begin{pmatrix} \vec{t} \\ \vec{p} \\ \vec{b} \end{pmatrix}$. Then,

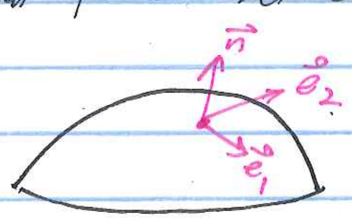
$$\dot{\Psi} = A \Psi, \text{ with } A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

Surfaces

Consider a surface in 3-Dimensional Euclidean space, parametrized by two coordinates x^i, $i=1,2$: $\vec{X}(x^1, x^2)$ specifies the pts on the surface.

Define the two vectors $\boxed{\vec{e}_i \equiv \partial_i \vec{X} = \frac{\partial \vec{X}}{\partial x^i}}$ — basis vectors on the surface.

Tangent plane: set of points $u^i \vec{e}_i$, for real u^i.



$$\vec{n} = \frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|}$$ unit normal to surface.

Distances on the surface:

$$d\vec{X} = \partial_i \vec{X}\, dx^i$$

$$ds^2 = d\vec{X} \cdot d\vec{X} = (\partial_i \vec{X} \cdot \partial_j \vec{X})\, dx^i dx^j$$

$$= \underbrace{\vec{e}_i \cdot \vec{e}_j}\, dx^i dx^j$$

Induced metric (induced by the embedding in 3D Euclidean space)

$$\boxed{g_{ij} = \vec{e}_i \cdot \vec{e}_j}$$

$\partial_j \vec{e}_i = \tilde{\Gamma}^k_{ij} \vec{e}_k + K_{ij} \vec{n}$ for some $\tilde{\Gamma}^k_{ij}$, K_{ij}.

$$\vec{n} \cdot \partial_j \vec{e}_i = \tilde{\Gamma}^k_{ij} \cancel{\vec{n} \cdot \vec{e}_k} + K_{ij}\, \vec{n} \cdot \vec{n}$$

$$\Rightarrow \boxed{K_{ij} = \vec{n} \cdot \partial_j \vec{e}_i}$$ Gauss's equation

$$\vec{e}_\ell \cdot \partial_j \vec{e}_i = \tilde{\Gamma}^k_{ij}\, \vec{e}_\ell \cdot \vec{e}_k + \cancel{K_{ij}\, \vec{e}_\ell \cdot \vec{n}}$$

$$= \tilde{\Gamma}^k_{ij}\, g_{k\ell}$$

Because $\partial_j \vec{e}_i = \partial_j \partial_i \vec{X} = \partial_i \partial_j \vec{X} = \partial_i \vec{e}_j$,
$\tilde{\Gamma}^k_{ij}$ and K_{ij} are symmetric in i,j.

$$\tilde{\Gamma}^k_{ij}\, g_{k\ell}\, g^{\ell m} = \boxed{\tilde{\Gamma}^m_{ij} = g^{\ell m}\, \vec{e}_\ell \cdot \partial_j \vec{e}_i}$$

Using $\partial_j g_{\ell i} = \vec{e}_i \cdot \partial_j \vec{e}_\ell + \vec{e}_\ell \cdot \partial_j \vec{e}_i$,

$$\boxed{\tilde{\Gamma}^m_{ij} = \frac{1}{2} g^{\ell m} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right)}$$ Exercise.

Hence, $\tilde{\Gamma}^m_{ij} = \Gamma^m_{ij}$, the Christoffel symbols.

The Γ^k_{ij} tell us how the tangent plane rotates about the normal vector on the surface, while K_{ij} tells us how the tangent plane moves in the embedding space.