Quantum + Gravity

Most of this lecture will not be about quantum gravity, but rather quantum fields in curved spacetime backgrounds. However, we begin with a few comments on quantum gravity.

Linearized Einstein gravity in vacuum can be quantized following the usual rules of quantum field theory. The theory describes noninteracting massless particles (gravitons) with helicity ± 2. This is a boring theory, but can be generalized to a quantum theory of linearized fluctuations about other backgrounds. This is the context in which the spectrum of gravity wave fluctuations during inflation is calculated.

Straightforward attempts to include the nonlinearities in a quantum theory of Einstein gravity have failed. An oversimplified explanation of this failure is related to a basic feature of renormalizable quantum field theories, namely that in natural units ($\hbar = c = 1$), all coupling constants are either dimensionless or have positive mass dimension. Renormalizable field theories are predictive at arbitrarily high energies; nonrenormalizable theories are not. In GR the coupling is $G_N \propto \frac{1}{M_{Pl}^2}$. G_N has negative mass dimension, so GR is nonrenormalizable as a quantum field theory.

Most attempts to define a quantum theory of gravity modify the notion of spacetime at short distances, and the macroscopic spacetime of GR emerges as an effective description at large distances. Dynamical triangulations and loop quantum gravity are examples. In string theory the elementary degrees of freedom are relativistic strings, not particles, and the graviton corresponds to a particular quantum fluctuation of the string. At present no attempt to quantize gravity is known to be consistent with the Standard Model of particle physics.

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What is a Particle?

Particles in Minkowski Spacetime

Consider a scalar field with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2}m^2\phi^2.$$

The Euler-Lagrange equation is the Klein-Gordon equation

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu \phi + m^2\phi = 0.$$

One set of solutions is

$$u_{\vec{k}}(t,\vec{x}) = \exp\left[i\vec{k}\cdot\vec{x} - i\omega t\right]\cdot\frac{1}{\sqrt{(2\pi)^3\cdot 2\omega}}$$

where

$$\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}, \qquad -\infty < k_i < \infty.$$

These modes are eigenfunctions of $i\frac{\partial}{\partial t}$,

$$i\frac{\partial}{\partial t}u_{\vec{k}}(t,\vec{x}) = \omega_{\vec{k}}u_{\vec{k}}(t,\vec{x}).$$

The eigenvalue $\omega_{\vec{k}}$ is called the (angular) frequency, and $\vec{k}$ is called the momentum.

A generic solution to the Euler-Lagrange equation can be expanded in the modes $u_{\vec{k}}$:

$$\phi(t,\vec{x}) = \sum_{\vec{k}}\left[a_{\vec{k}}\, u_{\vec{k}}(t,\vec{x}) + a_{\vec{k}}^{\dagger}\, u_{\vec{k}}^{*}(t,\vec{x})\right]$$

Canonical Quantization of the scalar field proceeds by imposing commutation relations analogous to
$$\begin{cases} [x, p] = i\hbar \\ [x, x] = [p, p] = 0. \end{cases}$$

The canonical momentum conjugate to ϕ is defined by analogy with $p = \frac{\partial L}{\partial \dot{q}}$ in particle mechanics:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$$

The canonical commutation relations then take the form

$$\begin{cases} [\phi(t,\vec{x}), \phi(t,\vec{x}')] = 0 \\ [\pi(t,\vec{x}), \pi(t,\vec{x}')] = 0 \\ [\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\,\delta^3(\vec{x}-\vec{x}') \end{cases}$$

By substituting the expansion for ϕ and π in terms of $u_{\vec{k}}(t,\vec{x})$, the canonical commutation relations become commutation relations for the coefficients $a_{\vec{k}}$ and $a_{\vec{k}}^{+}$, which are now operators:

$$\begin{cases} [a_{\vec{k}}, a_{\vec{k}'}] = 0 \\ [a_{\vec{k}}^{+}, a_{\vec{k}'}^{+}] = 0 \\ [a_{\vec{k}}, a_{\vec{k}'}^{+}] = \delta^3(\vec{k}-\vec{k}') \end{cases}$$

These resemble the harmonic oscillator commutators for annihilation and creation operators $a_{\vec{k}}$ and $a_{\vec{k}}^{+}$, respectively.

The vacuum is defined by the condition
$$a_{\vec{k}}|0\rangle = 0 \quad \text{for all } \vec{k}.$$

The state $|1_{\vec{k}}\rangle \equiv a_{\vec{k}}^{\dagger}|0\rangle$ is an eigenstate of the number operator $N \equiv \sum_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}}$ with eigenvalue 1.
The Hamiltonian for ϕ is $H = \pi \partial_t \phi - \mathcal{L}$ and can be written in terms of $a_{\vec{k}}$ and $a_{\vec{k}}^{\dagger}$:

$$H = \frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}} \left(a_{\vec{k}}^{\dagger} a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^{\dagger} \right).$$

Similarly the momentum is

$$\vec{P} = \sum_{\vec{k}} \vec{k}\, a_{\vec{k}}^{\dagger} a_{\vec{k}}$$

The state $|1_{\vec{k}}\rangle$ is an eigenstate of H and $\vec{P}$ with eigenvalues

$$E = \omega_{\vec{k}} + \sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} + \sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}}$$

zero-point energy.

$$\vec{P} = \vec{k}$$

The constant $\sum_{\vec{k}} \frac{1}{2}\omega_{\vec{k}}$ is the zero-point energy and contributes to every state the same amount.
Hence $|1_{\vec{k}}\rangle$ has the quantum numbers of a 1-particle state with momentum $\vec{k}$ and energy $\omega_{\vec{k}}$.
We call $|1_{\vec{k}}\rangle$ a 1-particle state.

Acting on the vacuum by additional creation operators $a^{+}_{\vec{k}_i}$ gives multiparticle states labeled by $\{\vec{k}_i\}$.

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Particles in curved spacetime

The notion of a particle is more subtle in a generic background spacetime. Which coordinate system should be used to define the frequency (what is $i\frac{\partial}{\partial t}$?) and to choose the complete set of modes $\{u_j(t,\vec{x})\}$?

Consider the Lagrangian density

$$\mathcal{L} = \sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2\right)$$

The Euler-Lagrange equation for ϕ is

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu\phi\right) + m^2\phi = 0$$

Suppose we find a complete set of modes $\{u_j, u_j^*\}$ and decompose ϕ as before:

$$\phi(t,\vec{x}) = \sum_j\left[a_j u_j(t,\vec{x}) + a_j^{+} u_j^*(t,\vec{x})\right]$$

with appropriately normalized $u_j(t,\vec{x})$, we impose the canonical commutation relations

$$[a_i, a_j^{+}] = \delta_{ij} \quad \text{, etc.}$$

The vacuum satisfies $a_i|0\rangle = 0$ for all i.

In Minkowski space the vacuum is invariant under Poincaré transformations. A generic spacetime does not have Poincaré invariance, so there is no reason to expect the vacuum to have any such invariance.

So, consider a second complete set of modes $\{v_j, v_j^*\}$ and decompose the field ϕ as

$$\phi(t,\vec{x}) = \sum_j \left[b_j v_j(t,\vec{x}) + b_j^\dagger v_j^*(t,\vec{x}) \right]$$

The vacuum defined by $b_j|\bar{0}\rangle = 0$ for all j in general does not coincide with the vacuum $|0\rangle$ defined by $a_j|0\rangle = 0$.

Since $\{u_j\}$ and $\{v_j\}$ are both complete sets, we can write

$$\left.\begin{aligned} v_j &= \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right) \\ u_i &= \sum_j \left(\alpha_{ji}^* v_j - \beta_{ji} v_j^* \right) \end{aligned}\right\} \text{Bogolubov transformation}$$

The creation and annihilation operators can also be related to one another in the two bases.

$$a_i = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger \right)$$

$$b_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \right)$$

The Bogolubov coefficients α_{ij}, β_{ij} satisfy the following:

$$\sum_k \left(\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^* \right) = \delta_{ij}$$

$$\sum_k \left(\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} \right) = 0$$

To preserve canonical commutation relations on b_i, b_j^+.

As long as $\beta_{ji} \neq 0$ the vacua and excitations of the vacua in the two bases disagree.

$$a_i |\bar{0}\rangle = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^+ \right) |\bar{0}\rangle$$

$$= \beta_{ji}^* |\bar{1}_j\rangle \neq 0$$

If $N_i = a_i^+ a_i$ is the number operator for the number of u_i-mode particles, then the expectation value for the number of u_i-mode particles in the state $|\bar{0}\rangle$ is

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2 .$$

$\Rightarrow$ The vacuum of the v_j modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode.

What Does a Particle Detector Detect?

Unruh and DeWitt modeled a particle detector as an idealized point particle with internal energy levels labeled by E, coupled to the scalar field ϕ by an interaction

$$\mathcal{L} = c\, m(\tau)\, \phi[x^\mu(\tau)]\, \delta^3(\vec{x} - \vec{x}(\tau))$$

where $x^\mu(\tau)$ describes the trajectory of the point particle and τ is the particle's proper time. $m(\tau)$ = monopole operator

Suppose the field ϕ is in the Minkowski vacuum $|0_M\rangle$, in the Minkowski spacetime. Depending on the trajectory $x^\mu(\tau)$, it is possible that the detector will transition to an excited state with energy $E > E_0$ (E_0 = ground state energy), while the field transitions to a state $|\Psi\rangle$.

Perturbation theory: transition amplitude $\mathcal{A}$ is

$$\mathcal{A} = ic \langle E, \Psi | \int_{-\infty}^{\infty} d\tau\, m(\tau)\, \phi[x^\mu(\tau)]\, |0_M, E_0\rangle$$

Heisenberg picture evolution of $m(\tau)$: $m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}$,
$H_0 |E\rangle = E|E\rangle$

$$\mathcal{A} = ic \langle E| m(0) |E_0\rangle \int_{-\infty}^{\infty} d\tau\, e^{i(E-E_0)\tau} \langle \Psi | \phi(x) | 0_M\rangle$$

$\mathcal{A} \neq 0$, if $|\Psi\rangle = |1_{\vec{k}}\rangle$:
(possibly)

$$\langle 1_{\vec{k}} | \phi(x) | 0_M \rangle = \frac{1}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \exp\left(-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t\right)$$

$$\Rightarrow a_{\vec{k}} = \frac{ic \langle E | m(0) | E_0 \rangle}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \int_{-\infty}^{\infty} e^{i(E-E_0)\tau} e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t} d\tau$$

<u>Example:</u> Inertial particle detector

$$\vec{x} = \vec{x}_0 + \vec{v}t = \vec{x}_0 + \frac{\vec{v}\tau}{\sqrt{(1-v^2)}}$$

$$a_{\vec{k}} = \frac{ic \langle E | m(0) | E_0 \rangle}{\sqrt{16\pi^3 \omega_{\vec{k}}}} \cdot 2\pi\, \delta\left(E - E_0 + \frac{(\omega - \vec{k}\cdot\vec{v})}{\sqrt{1-v^2}}\right)$$

But $E > E_0$ and $\vec{k}\cdot\vec{v} \leq |\vec{k}||\vec{v}| < \omega$

$$\Rightarrow E - E_0 + \frac{(\omega - \vec{k}\cdot\vec{v})}{\sqrt{1-v^2}} > 0 \longrightarrow \delta\text{-fn vanishes}$$

$$\rightarrow a_{\vec{k}} = 0$$

☆ $\Rightarrow$ An inertial particle detector in the Minkowski spacetime vacuum detects no particles.

Example : Noninertial detector trajectory.

Transition probability to all possible E and Ψ

$$P = \sum_E |a_{\Psi,E}|^2 = c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \mathcal{F}(E-E_0)$$

where $$\mathcal{F}(E) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \, e^{-iE(\tau-\tau')} \underbrace{\langle 0_M|\phi(x^\mu(\tau))\,\phi(x^\mu(\tau'))|0_M\rangle}_{G^+(x(\tau),\,x(\tau'))}$$

$\equiv$ positive-frequency Wightman function.

$\mathcal{F}(E)$ = detector response function
— represents the bath of particles detected by the detector as a result of its motion.

$c^2|\langle E|m(0)|E_0\rangle|^2$ factor describes selectivity of detector to the bath — depends on details of detector.

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Example: massless field ϕ, $m^2=0$, $z=(t^2+\alpha^2)^{1/2}$ trajectory (constant α^{-1} = acceleration)
$G^+(x,x')$ can be calculated, then $\mathcal{F}(E)$ determined.

Result:
$$P = \frac{c^2}{2\pi} \sum_E \frac{(E-E_0)\,|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)\alpha}-1}$$

This is equivalent to what the particle detector would have detected if unaccelerated but in a bath of particles at temperature $T = \frac{1}{2\pi\alpha k_B} = \frac{\text{acceleration}}{2\pi k_B}$

k_B = Boltzmann's const.

* This is the main result, known as the Unruh effect: An accelerated particle detector in Minkowski spacetime (Rindler spacetime) detects a finite-temperature distribution of particles with temperature proportional to the acceleration.

Another way to understand this result, which we will not explain here, is in terms of the particle horizon of the accelerated observer. (Recall from our earlier discussion of Rindler spacetime that Rindler coordinates cover only part of Minkowski spacetime.)

A related result is that an observer far from a black hole observes a finite-temperature distribution of particles with temperature $\underbrace{T_H}_{\text{Hawking temperature}} = \frac{1}{8\pi G M k_B}$

(Temperatures in units where $\hbar = c = 1$. For example,
$$T_H = \frac{\hbar c^3}{8\pi G M k_B}.$$)

Another related result is that particles can be created in a non-static spacetime.

To Summarize: Gravity + Quantum = Interesting!

A few comments about black holes

The Hawking temperature could have been determined up to an overall constant on dimensional grounds. In particular, $\boxed{T_H \propto \frac{1}{GM}}$.

As the black hole radiates, M decreases and T_H increases. Black hole decay is explosive!

In the 1970s Carter, Bekenstein, Hawking and a few others pointed out the analogies between black hole mechanics and thermodynamics, even before Hawking's discovery of Hawking radiation. The first law of thermodynamics allows us to define the entropy of a black hole:

$$dMc^2 = T\,dS = \frac{\hbar c^3}{8\pi GMk_B}\,dS$$

$$\rightarrow S \cdot \frac{\hbar c}{8\pi G k_B} = \frac{M^2}{2} = \left(\frac{c^2 R_S}{2G}\right)^2 \cdot \frac{1}{2}$$

$$S = \frac{c^3 k_B}{4G\hbar} \cdot \left(4\pi R_S^2\right), \quad \text{i.e.}$$

$$\boxed{S = \frac{c^3 k_B}{4G\hbar} A_H}, \quad \text{where } A_H = \text{horizon area}$$

The entropy of the black hole is proportional to the horizon area. Entropy is not an extensive quantity in this context! It grows as an area, not a volume!

This is the motivation for the notion of <u>holography</u> in quantum systems coupled to gravity. In 1993 't Hooft suggested that in the presence of gravity, the # degrees of freedom around a black hole is that of a system of one dimension smaller, i.e. an area rather than a volume. In 1995 Susskind generalized this idea to conjecture that any quantum system with gravity effectively has the number of degrees of freedom of a nongravitational system with one dimension fewer.

In 1997, Maldacena discovered an explicit example of Susskind's holographic principle in string theory. This realization is known as the AdS/CFT correspondence.

<u>The Firewall Puzzle</u>

In 2012 Almheiri, Marolf, Polchinski and Sully (AMPS) discovered a puzzle related to black holes radiation that has received a lot of attention since then. The puzzle is roughly the following:

1) A radiated particle involves two entangled particles: one outgoing and one infalling (particle-antiparticle production near the horizon).
2) The radiated particle is entangled with the previously radiated particles (in order for the black hole to be in a pure quantum state). (Susskind, Thorlacius; 't Hooft)

But it is a basic theorem in quantum mechanics that no particle can be maximally entangled with two independent systems at the same time. So (1) and (2) can't both be correct.

One possibility is that the entanglement of the outgoing radiated particle and the infalling particle is broken. The non-entangled state would have higher energy, and the horizon of the black hole would be a hot "firewall," which would prevent freely falling observers from entering the black hole. This would contradict the equivalence principle because the observer falling into the black hole would be able to distinguish the system from an inertial system in empty flat space.

Hawking has argued that because black holes evaporate there is no real event horizon, and that no firewall is required to be consistent with black hole thermodynamics.

Hossenfelder has argued that the radiated particles are not in a pure state, so again there is no firewall.

The firewall puzzle continues to be debated, and black holes continue to provide fodder for curious folks.