The Schwarzschild Horizon

The Schwarzschild metric is singular at $r_s = 2GM$, where $g_{tt} = 0$, $g_{rr} = \infty$.
The gravitational redshift and time dilation become divergent as $r \to 2GM$. The value $r_s = 2GM$ is called the Schwarzschild radius, and the surface $r = 2GM$ is called the Schwarzschild horizon.

Light travels along <u>null geodesics</u> with $ds^2 = 0$.
A trajectory with $\theta, \varphi = $ const. satisfies

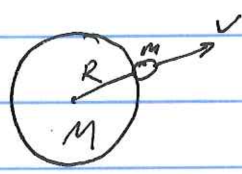
$$\frac{dr}{dt} = \left(1 - \frac{2GM}{r}\right) \to 0 \text{ as } r \to 2GM.$$

The Schwarzschild horizon separates causally distinct regions in the spacetime, $r < 2GM$ and $r > 2GM$.
$\Rightarrow$ Schwarzschild is a <u>black hole</u> spacetime.
Note that the Schwarzschild solution is only relevant where there is no matter so $T_{\mu\nu} = 0$. If matter is not compressed within the Schwarzschild radius then the geometry will be regular.

Typical Schwarzschild radii:
Sun: $2GM_\odot = 2.95$ km
proton: $2GM_p = 10^{-52}$ m
Earth: $2GM_E = 8.87 \times 10^{-3}$ m

It is important to note that despite the physical importance of the Schwarzschild horizon, the spacetime curvature $R_{\mu\nu\rho\sigma}$ is nonsingular at $r=2GM$. An observer falling through the horizon will feel no infinite stresses at the horizon.

It is amusing to compare the Schwarzschild radius to the size of an object whose Newtonian escape velocity is the speed of light c.



$$\frac{1}{2} m v_{escape}^2 - \frac{GMm}{R} = 0$$

$$v_{escape}^2 = c^2 = \frac{2GM}{R}$$

$$R = \frac{2GM}{c^2} = R_s$$

The Schwarzschild geometry approximates the spacetime in the neighborhood of the sun, so we will analyze trajectories which describe planetary motion and bending of light by the sun.

Coordinate Singularities

Wald 6.4

In the Schwarzschild spacetime curvature invariants like $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ behave regularly around $r=2GM$, but the metric depends on $\left(1-\frac{2GM}{r}\right)^{1/2}$ which behaves singularly.

This is an example of a coordinate singularity, which can be eliminated by a change of coordinates. A simple analogy is provided by the metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2 , \qquad \begin{array}{c} -\infty < x < \infty \\ 0 < t < \infty \end{array}$$

A change of coordinates $t \to t' = 1/t$ removes the singularity at $t=0$:

$$ds^2 = (t')^4\left(\frac{-dt'}{(t')^2}\right)^2 + dx^2$$

$$= -(dt')^2 + dx^2 \qquad \text{Minkowski space.}$$

The region covered by the original coordinates $0<t<\infty$ is the upper-half plane in Minkowski space, $0<t'<\infty$.

The spacetime described by the original metric is <u>geodesically complete</u> as $t \to 0$, meaning all geodesics approaching $t=0$ extend to arbitrarily large values of their affine parameter τ.

However, geodesics may reach $t=\infty$ for finite values of their affine parameter, so the coordinates (x,t) do not describe a geodesically complete spacetime.

On the other hand, the geometry described by (x,t') may be made geodesically complete by extending the coordinate range from $0<t'<\infty$ to $-\infty<t'<\infty$.

Another example, similar in some respects to the Schwarzschild case, is the <u>Rindler Spacetime</u>,

$$ds^2 = -x^2 dt^2 + dx^2 \,, \quad \begin{cases} -\infty < t < \infty \\ 0 < x < \infty \end{cases}$$

The metric appears singular at $x=0$. Geodesics terminate with finite affine parameter at $x=0$, but the curvature is regular as $x\to 0$. Indeed, $R_{\mu\nu\lambda\sigma}=0$ everywhere in the spacetime.

Null geodesics: $-x^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = 0$

← Affine parameter, not proper time

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$t = \pm \ln x + \text{const.}$$

↑ + = "outgoing"
− = "ingoing"

Define coordinates

$$u = t - \ln x \,, \quad -\infty < u < \infty$$
$$v = t + \ln x \,, \quad -\infty < v < \infty$$

$v-u = 2\ln x$

$x^2 = e^{v-u}$

$v+u = 2t \longrightarrow t = \frac{v+u}{2}$

$dx = \frac{1}{2} e^{\frac{v-u}{2}} dv - \frac{1}{2} e^{\frac{v-u}{2}} du$

$dt = \frac{1}{2} dv + \frac{1}{2} du$

$ds^2 = -x^2 dt^2 + dx^2$

$= -\frac{e^{v-u}}{4}\left[dv^2 + du^2 + 2du\,dv\right] + \frac{e^{v-u}}{4}\left[dv^2 + du^2 - 2du\,dv\right]$

$\boxed{ds^2 = -e^{v-u}\,du\,dv}$, $\begin{cases} -\infty < u < \infty \\ -\infty < v < \infty \end{cases}$

Define $U = -e^{-u}$, $V = e^{v}$, $\begin{cases} -\infty < U < 0 \\ 0 < V < \infty \end{cases}$

$\boxed{ds^2 = -dU\,dV}$. Extending the coordinate range to $-\infty < U < \infty$, $-\infty < V < \infty$, the spacetime becomes geodesically complete.

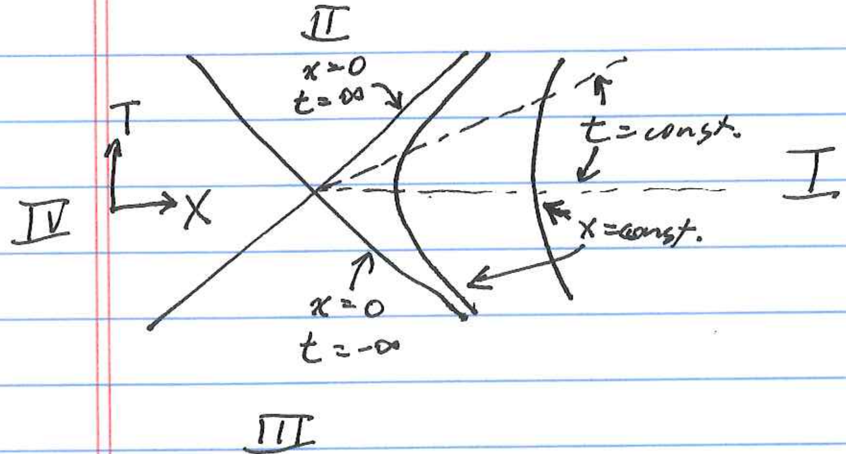
This geometry is just Minkowski space!

Define $T = \frac{U+V}{2}$, $X = \frac{V-U}{2}$

$\boxed{ds^2 = -dT^2 + dX^2}$, $\begin{cases} -\infty < T < \infty \\ -\infty < X < \infty \end{cases}$

In terms of the original coordinates,

$$\boxed{\begin{aligned} x &= (X^2 - T^2)^{1/2} \\ t &= \tanh^{-1}(T/X) \end{aligned}}$$



Rindler spacetime is the region I $(X > |T|)$ of Minkowski spacetime.

Consider a (non geodesic) trajectory $x = \text{const.}$ in the original coordinates.

The proper acceleration is $\boxed{a^\mu = \frac{D}{D\tau}\left(\frac{dx^\mu}{d\tau}\right) = U^\nu D_\nu U^\mu}$
where $U^\mu = \frac{dx^\mu}{d\tau}$,
and $U^\mu U^\nu g_{\mu\nu} = -1 \Longrightarrow U^\mu = \left(\frac{1}{x}, 0\right)$.
The nonvanishing Christoffel symbols are

$$\Gamma^t_{xt} = \Gamma^t_{tx} = -1/x$$

$$\Gamma^x_{tt} = x$$

$$a^\mu = U^\nu\left(\partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda\right) = \frac{dU^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} U^\nu U^\lambda$$
$$= (U^t)^2 \Gamma^\mu_{tt} = \frac{1}{x^2}\Gamma^\mu_{tt}$$

$$\boxed{a^x = \frac{1}{x}}, \quad a^t = 0$$

$\longrightarrow$ The Rindler coordinates (x, t) describe Minkowski space in an accelerated coordinate system.

Kruskal Coordinates

Consider the r, t part of the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

Null geodesics satisfy (with angular coordinates fixed),

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 0$$

Affine parameter, not proper time.

$$\Longrightarrow \left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-2}$$

Solutions: $\boxed{t = \pm r_* + \text{constant},}$

where $\boxed{r_* = r + 2GM \ln\left(r/2GM - 1\right)}$

is the "Regge-Wheeler tortoise coordinate."

Note that $\frac{dr_*}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1}$.

Define null coordinates u, v:

$$u = t - r_*$$
$$v = t + r_*$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2GM}{r}\right) du\, dv \quad , \text{ where } r = r(u,v)$$

from $r_* = \frac{v-u}{2}$

$$\rightarrow r + 2GM \ln\left(\frac{r}{2GM} - 1\right) = \frac{v-u}{2}$$

$$\frac{r}{2GM} - 1 = e^{(v-u)/4GM}\, e^{-r/2GM}$$

$$= \frac{r}{2GM}\left(1 - \frac{2GM}{r}\right)$$

$$\rightarrow ds^2 = -\underbrace{\frac{2GM\, e^{-r/2GM}}{r}}_{\text{Nonsingular as } r\to 2GM,\ (u\to\infty \text{ or } v\to -\infty)} e^{(v-u)/4GM}\, du\, dv$$

Define new coordinates $U = -e^{-u/4GM}$

$V = e^{v/4GM}$

$$\Rightarrow ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r}\, dU\, dV$$

No singularity at $r=2GM$ $(U=0 \text{ or } V=0)$

Change coordinates to $T = \frac{U+V}{2}$, $X = \frac{V-U}{2}$, and restore the angular coordinates:

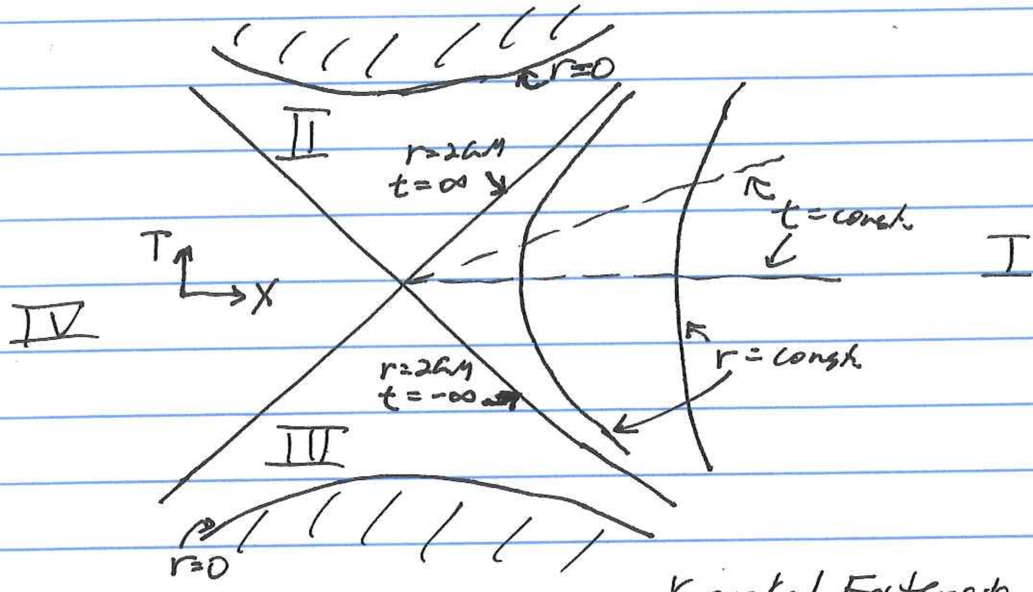
$$\boxed{ds^2 = \frac{32(GM)^3 e^{-r(u,v)/2GM}}{r(u,v)}\left(-dT^2 + dX^2\right) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

— Schwarzschild metric in Kruskal coordinates

$$\boxed{\begin{aligned} \left(\frac{r}{2GM} - 1\right) e^{r/2GM} &= X^2 - T^2 \\ \frac{t}{2GM} &= \ln\left(\frac{T+X}{X-T}\right) = 2\tanh^{-1}\left(T/X\right) \end{aligned}}$$

The singularity at $r=0$ is a true curvature singularity, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$ as $r\to 0$. The allowed range of coordinates X, T follows from the condition $r>0$

$$\Rightarrow \boxed{X^2 - T^2 > -1}$$



Kruskal Extension of Schwarzschild Spacetime

Region I : $r > 2GM$ of original Schwarzschild spacetime.

Region II : Black Hole - all observers in this region reach the singularity at $r=0$ in finite proper time.

Region III : White Hole - all observers originated at $r=0$ $(X = -(T^2-1)^{1/2})$ and leave region III in finite proper time.

Region IV : looks like Region I - asymptotically flat, $r > 2GM$.

$t=0$.

Each circle represents a 2-sphere.