# The Schwarzschild Metric

Weinberg 8.1–8.2
Zee VI.1

Consider a spacetime metric of the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

(static, isotropic)

i.e.

$$g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2\sin^2\theta, \quad g_{tt} = -B(r)$$

$$g^{rr} = A^{-1}(r), \quad g^{\theta\theta} = r^{-2}, \quad g^{\varphi\varphi} = r^{-2}(\sin\theta)^{-2}, \quad g^{tt} = -B^{-1}(r)$$

We calculate the affine connection:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})$$

Nonvanishing components:

$$\Gamma^r_{rr} = \frac{1}{2}A^{-1}(r)\left(A'(r) + A'(r) - A'(r)\right) = \frac{1}{2}A^{-1}(r)\,A'(r)$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}A^{-1}(r)\left(-\frac{d}{dr}(r^2)\right) = -A^{-1}(r)\, r$$

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2}A^{-1}(r)\left(-\frac{\partial}{\partial r}(r^2\sin^2\theta)\right) = -A^{-1}(r)\, r\sin^2\theta$$

$$\Gamma^r_{tt} = \frac{1}{2}A^{-1}(r)\,B'(r), \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = -\frac{1}{2}B^{-1}(r)\,(-B'(r)) = \frac{1}{2}B^{-1}(r)\,B'(r)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2}r^{-2}\frac{d}{dr}(r^2) = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2}r^{-2}\left(-\frac{\partial}{\partial\theta}(r^2\sin^2\theta)\right) = -\sin\theta\cos\theta$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{2}(r^2\sin^2\theta)^{-1}\partial_r(r^2\sin^2\theta) = r^{-1}$$

$$\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \frac{1}{2}(r^2\sin^2\theta)^{-1}\frac{\partial}{\partial\theta}(r^2\sin^2\theta) = \cot\theta$$

The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\lambda} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\lambda\eta}$$

$$R_{rr} = \frac{d}{dr}\left( \cancel{\frac{A'(r)}{2A(r)}} + \frac{B'(r)}{2B(r)} + \cancel{\frac{1}{r} + \frac{1}{r}} \right) - \frac{d}{dr}\left( \cancel{\frac{A'(r)}{2A(r)}} \right)$$

$$+ \left( \cancel{\left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2} + \left(\frac{A'}{2A}\right)^2 + \left(\frac{B'}{2B}\right)^2 \right)$$

$$- \left(\frac{A'}{2A}\right)\left( \frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r} \right)$$

$$= \frac{B''}{2B} - \frac{1}{2B^2}(B')^2 + \cancel{\frac{1}{4A^2}(A')^2} + \frac{1}{4B^2}(B')^2$$

$$- \cancel{\frac{1}{4A^2}(A')^2} - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

$$= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r}\left( \frac{A'}{A} \right)$$

Similarly,

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left( -\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A}$$

$$R_{\varphi\varphi} = \sin^2\theta \left( -1 + \frac{r}{2A}\left( -\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} \right)$$

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\left(\frac{B'}{A}\right)\left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r}\left(\frac{B'}{A}\right)$$

$R_{\mu\nu} = 0$ for $\mu \neq \nu$.

The vacuum Einstein eqs are $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.

Contracting $\mu, \nu$: $R - \frac{1}{2} \cdot 4R = -R = 0$

$$\Rightarrow \boxed{R_{\mu\nu} = 0} \quad \text{in vacuum.}$$

Use $\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$

$$\Rightarrow \frac{B'}{B} = -\frac{A'}{A} \Longrightarrow A(r)\, B(r) = \text{const.}$$

As $r \to \infty$ we will insist that the metric approach the Minkowski metric in spherical coordinates,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

$$\longrightarrow \boxed{A(r) = {}^{1}/_{B(r)}}$$

$$R_{\theta\theta} = -1 + \frac{r}{2} B\left(-B \frac{d}{dr}({}^{1}/_{B}) + {}^{B'}/_{B}\right) + B = 0$$

$$= -1 + rB' + B \quad = 0$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\left(\frac{B'}{B}\right)\cancel{\left(B \frac{d}{dr}({}^{1}/_{B}) + {}^{B'}/_{B}\right)} - \frac{1}{r}\left(B \frac{d}{dr}({}^{1}/_{B})\right)$$

$$= \frac{B''}{2B} + \frac{B'}{rB} = \frac{1}{2rB} \frac{d}{dr}(R_{\theta\theta})$$

So, if $R_{\theta\theta} = 0$ everywhere, then $R_{rr} = 0$ everywhere.

$$R_{\theta\theta} = \frac{d}{dr}(r B(r)) - 1 \quad = 0$$

$$\Rightarrow r B(r) = r + \text{const.}$$

To fix the constant, we will assume that at large $r$, the metric agrees with the Newtonian limit,

$$g_{tt} = -B \xrightarrow[r\to\infty]{} -1-2\phi \;, \text{ where}$$

$(G \equiv G_N)$

$\phi = -\frac{GM}{r}$ is the potential due to a point mass $M$ at $r=0$. (But note that there is no mass sitting at $r=0$ in the Schwarzschild spacetime.)

Hence, $B(r) = 1 - \frac{2GM}{r}$

$$A(r) = \left(1-\frac{2GM}{r}\right)^{-1}$$

$$\boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

Schwarzschild Metric (1916) in Standard Form.

Defining a new radius variable

$$r \equiv \rho\left(1+\frac{GM}{2\rho}\right)^2 ,$$

Exercise.

$$\boxed{ds^2 = \frac{-(1-GM/2\rho)^2}{(1+GM/2\rho)^2}\,dt^2 + \left(1+\frac{GM}{2\rho}\right)^4\left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)}$$

Schwarzschild metric in Isotropic Form.

To examine the energy (mass) of the black hole we need a definition of energy in a gravitational field.

Weinberg 7.6

## Energy-Momentum Tensor of Gravitation

Consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \to 0$ as $x^\mu \to \infty$, but $h_{\mu\nu}$ is not necessarily small.

The part of $R_{\mu\nu}$ linear in $h_{\mu\nu}$ is

$$R^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right)$$

where indices on $h_{\mu\nu}$, $R^{(1)}_{\mu\nu}$, and $\frac{\partial}{\partial x^\lambda}$ are raised and lowered by $\eta_{\alpha\beta}$, not $g_{\alpha\beta}$. For example, $h^\lambda{}_\lambda \equiv \eta^{\lambda\nu} h_{\nu\lambda}$

True tensors like $R_{\mu\nu}$ will continue to have indices raised and lowered by $g_{\mu\nu}$.

Einstein equations:

$$\boxed{R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_\lambda = -8\pi G_N \left(T_{\mu\nu} + t_{\mu\nu}\right)}$$

where $\boxed{t_{\mu\nu} = \frac{1}{8\pi G_N}\left\{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^\lambda{}_\lambda - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_\lambda\right\}}$

Einstein's equations have the form of a wave-like equation for $h_{\mu\nu}$ with source $T_{\mu\nu} + t_{\mu\nu}$.
IF $T_{\mu\nu}$ is the energy-momentum tensor of matter, it is natural to consider $t_{\mu\nu}$ the energy-momentum "tensor" of gravitation. ($t_{\mu\nu}$ does not transform as a tensor.)

The *total* energy-momentum tensor of matter and gravitation is $\boxed{\tau_{\mu\nu} \equiv T_{\mu\nu} + t_{\mu\nu}}$.

Properties of $\tau_{\mu\nu}$:

1) $\tau_{\mu\nu} = \tau_{\nu\mu}$ $\qquad \tau^{\nu\lambda} \equiv \eta^{\nu\sigma}\eta^{\lambda\rho}\,\tau_{\sigma\rho}$

2) $\boxed{\dfrac{\partial}{\partial x^\nu}\tau^{\nu\lambda} = 0}$, i.e. $\tau^{\nu\lambda}$ is *conserved* in the ordinary sense. (by the linearized Bianchi ID, $\partial_\mu(R^{\mu\nu(1)} - \frac{1}{2}\eta^{\mu\nu}R^{(1)}) \equiv 0$)

Hence, $P^\lambda = \int d^3x\, \tau^{0\lambda}$ has the interpretation of the energy-momentum "vector" of the system, including gravitation. — Not generally covariant, but Lorentz covariant.

3) Expanding in $h_{\mu\nu}$, the first term in $t_{\mu\nu}$ is quadratic:

$$t_{\mu\nu} = \frac{1}{8\pi G}\Big[-\frac{1}{2}h_{\mu\nu}R^{(1)\lambda}{}_\lambda + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\Big] + O(h^3)$$

where

$$R^{(2)}_{\mu\nu} = -\frac{1}{2}h^{\lambda\kappa}\Big[\partial_\nu\partial_\mu h_{\lambda\kappa} - \partial_\nu\partial_\lambda h_{\mu\kappa} - \partial_\kappa\partial_\mu h_{\lambda\nu} + \partial_\kappa\partial_\lambda h_{\mu\nu}\Big]$$
$$+ \frac{1}{4}\Big[2\partial_\kappa h^\kappa{}_\sigma - \partial_\sigma h^\kappa{}_\kappa\Big]\Big[\partial_\nu h^\sigma{}_\mu + \partial_\mu h^\sigma{}_\nu - \partial^\sigma h_{\mu\nu}\Big]$$
$$- \frac{1}{4}\Big[\partial_\lambda h_{\sigma\nu} + \partial_\nu h_{\sigma\lambda} - \partial_\sigma h_{\lambda\nu}\Big]\Big[\partial^\lambda h^\sigma{}_\mu + \partial_\mu h^{\sigma\lambda} - \partial^\sigma h^\lambda{}_\mu\Big]$$

We can calculate the total energy and momentum in any system whose metric outside some region is given by the Schwarzschild metric.

Weinberg 7.6

Write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The energy-momentum tensor of matter + gravitation is

$$-8\pi G_N T^{\mu\nu} = R^{(1)\mu\nu} - \frac{1}{2}\eta^{\mu\nu} R^{(1)}$$
$$= \frac{1}{2}\Big(\partial^\mu \partial^\nu h^\lambda{}_\lambda - \partial_\lambda \partial^\nu h^{\lambda\mu} - \partial_\lambda \partial^\mu h^{\lambda\nu} + \partial_\lambda \partial^\lambda h^{\mu\nu}$$
$$- \eta^{\mu\nu}\partial_\lambda \partial^\lambda h^\rho{}_\rho + \eta^{\mu\nu}\partial_\lambda \partial_\rho h^{\lambda\rho}\Big)$$

where indices here are contracted with $\eta_{\alpha\beta}$.
This can be written as a divergence:

$$\boxed{-8\pi G_N T^{\mu\nu} = \partial_\rho Q^{\rho\mu\nu}}, \text{ where}$$

$$\boxed{Q^{\rho\nu\lambda} = \frac{1}{2}\Big\{\partial^\nu h^\mu_\mu \eta^{\rho\lambda} - \partial^\rho h^\mu{}_\mu \eta^{\nu\lambda} - \partial_\mu h^{\mu\nu}\eta^{\rho\lambda} + \partial^\mu h_\mu{}^\rho \eta^{\nu\lambda} + \partial^\rho h^{\nu\lambda} - \partial^\nu h^{\rho\lambda}\Big\}}$$

Note that $Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda}$. (Consider the linearized Bianchi identity $\partial_\mu T^{\mu\nu} = 0$.)

Then $P^\lambda = -\frac{1}{8\pi G_N}\int_V \partial_\rho Q^{\rho 0\lambda}\, d^3x = -\frac{1}{8\pi G_N}\int \partial_i Q^{i0\lambda}\, d^3x$

$$\overset{\text{Gauss' theorem}}{=} -\frac{1}{8\pi G_N}\int Q^{i0\lambda}\, n_i\, r^2\, d\Omega \text{ , where}$$

$r \equiv \sqrt{x^i x^i}$, $n_i \equiv x^i/r$, $d\Omega = \sin\theta\, d\theta\, d\varphi$

The total energy is

$$P^0 = -\frac{1}{8\pi G_N}\int Q^{i00}\, n^i r^2\, d\Omega$$

$$= -\frac{1}{16\pi G_N}\int \left\{ -\frac{\partial}{\partial x^i} h^\mu{}_\mu (-1) + \frac{\partial}{\partial x^\mu} h^{\mu i} (-1) + \frac{\partial}{\partial x^i} h^{00} \right\} n^i r^2\, d\Omega$$

$$\boxed{P^0 = -\frac{1}{16\pi G_N}\int \left\{ \frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j} \right\} n^i r^2\, d\Omega}$$

$$P^j = -\frac{1}{8\pi G_N}\int Q^{i0j}\, n^i r^2\, d\Omega$$

$$\boxed{P^j = -\frac{1}{16\pi G_N}\int \left\{ -\frac{\partial}{\partial t} h_{kk}\delta_{ij} + \frac{\partial}{\partial x^k} h_{k0}\delta_{ij} - \frac{\partial}{\partial x^i} h_{j0} + \frac{\partial}{\partial t} h_{ij} \right\} n^i r^2\, d\Omega}$$

For the Schwarzschild metric, $h_{k0} = 0$ and $\frac{\partial}{\partial t} h_{ij} = 0$, so $P^j = 0$. This is not surprising: the static, isotropic (same in every direction) Schwarzschild solution has no spatial momentum.

It is convenient to define quasi-Minkowskian coordinates:

$$x^1 \equiv r\sin\theta\cos\varphi,\quad x^2 \equiv r\sin\theta\sin\varphi,\quad x^3 \equiv r\cos\theta$$

where $r, \theta, \varphi$ are the standard Schwarzschild coords. The metric becomes,

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left\{\left(1 - \frac{2GM}{r}\right)^{-1} - 1\right\} r^{-2}\left(x^i dx^i\right)^2 + dx^i dx^i}$$

where $r = \sqrt{x^i x^i}$, $\quad dr = \frac{1}{r} x^i dx^i$

$$h_{ij} \equiv g_{ij} - \delta_{ij} \xrightarrow[r\to\infty]{} \frac{2GM}{r} n_i n_j + \mathcal{O}(1/r^2)$$

where $n_i \equiv x^i/r$ , $n_i n_i = 1$

Use $\frac{\partial r}{\partial x^i} = \frac{1}{r} x^i = n_i$

$$\frac{\partial n_i}{\partial x^j} = \frac{1}{r}\delta_{ij} - \frac{1}{r^2} x^i x^j = \frac{1}{r}(\delta_{ij} - n_i n_j)$$

$$\Rightarrow \frac{\partial}{\partial x^i} h_{jj} - \frac{\partial}{\partial x^j} h_{ij} \xrightarrow[r\to\infty]{} \frac{\partial}{\partial x^i}\left(\frac{2GM}{r}\right) - \frac{\partial}{\partial x^j}\left(\frac{2GM}{r} n_i n_j\right)$$

$$= -\frac{2GM}{r^2} n_i - \left\{ -\frac{2GM}{r^2} n_j n_i n_j + \frac{2GM}{r^2}(\delta_{ij} - n_i n_j) n_j + \frac{2GM}{r^2} n_i \underbrace{(\delta_j{}^j - n_j n_j)}_{2} \right\}$$

$$= -\frac{4GM}{r^2} n_i$$

$$\Rightarrow P^0 = -\frac{1}{16\pi G} \int \left(-\frac{4GM}{r^2}\right) n_i n_i \, r^2 \, d\Omega$$

$$\boxed{P^0 = M}$$

Hence, the total energy of matter + gravitation in any spacetime which is described by the Schwarzschild metric outside some region is $M$.