Weinberg 7.1, Zee VI.1

# Einstein's Field Equations

In the absence of gravitation the energy-momentum tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$.
By the principle of general covariance, we expect the conservation law to be covariant, so that the energy-momentum tensor is instead covariantly conserved,

$$\boxed{D_\mu T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\mu} = 0}$$

In the nonrelativistic, weak field limit, $T_{00}$ is the mass density $\rho$: $T_{00} \approx \rho$.
We have seen that in this limit $g_{00} \approx -(1+2\phi)$, where $\phi$ is the Newtonian gravitational potential.

The Poisson eqn. for $\phi$ is $\nabla^2\phi = 4\pi G_N \rho$, where $G_N = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$ is Newton's constant.

Combining the above, $\nabla^2 g_{00} \approx -8\pi G_N T_{00}$

It is natural to guess that the covariant form of this equation will take the form

$$\boxed{G_{\mu\nu} = -8\pi G_N T_{\mu\nu}}$$

for some tensor $G_{\mu\nu}$ such that $G_{00} \approx \nabla^2 g_{00}$ in the nonrelativistic limit.

We look for a tensor $G_{\mu\nu}$ such that:

(1) $G_{\mu\nu}$ consists of terms with 2 derivatives of $g_{\mu\nu}$.

(2) $G_{\mu\nu}$ is symmetric in $\mu \Leftrightarrow \nu$ (because $T_{\mu\nu}$ is).

(3) $D_\mu G^\mu{}_\nu = 0$ (because $D_\mu T^\mu{}_\nu = 0$).

(4) $G_{00} \approx \nabla^2 g_{00}$ in the nonrelativistic weak-field limit.

The Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$ is the only tensor that can be formed from the metric and its 1st or 2nd derivatives. Hence, the most general tensor satisfying (1) and (2) is made of contractions of $R_{\mu\nu\lambda\sigma}$:

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R$$

for some constants $c_1$ and $c_2$.

Recall the Bianchi identity, $D_\mu R^\mu{}_\nu = \frac{1}{2} D_\nu R$.

By condition (3), recall $D^\mu g_{\mu\nu} = 0$

$$0 = D_\mu G^\mu{}_\nu = c_1 D_\mu R^\mu{}_\nu + c_2 g_{\mu\nu} D^\mu R$$

$$= \left(\frac{c_1}{2} + c_2\right) D_\nu R$$

Hence, either $c_2 = -c_1/2$, or $D_\nu R = 0$ everywhere

But $G^\mu{}_\mu = (c_1 + 4c_2) R = -8\pi G T^\mu{}_\mu$

and $D_\nu T^\mu{}_\mu \neq 0$ in general, so

$$G_{\mu\nu} = c_1 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$$

Condition (4) then determines $c_1$.

For nonrelativistic systems $|T_{ij}| \ll |T_{00}|$, so $|G_{ij}| \ll |G_{00}|$.

$\rightarrow \quad R_{ij} \approx \frac{1}{2} g_{ij} R$

with $g_{\mu\nu} \approx \eta_{\mu\nu}$,

$$R \approx \sum_{k=1}^{3} R_{kk} - R_{00} \approx \frac{3}{2} R - R_{00}$$

$$\Rightarrow \quad R \approx 2 R_{00}$$

Then
$$G_{00} = C_1 \left( R_{00} - \frac{1}{2} g_{00} R \right)$$
$$\approx 2 C_1 R_{00}$$

$$R_{00} = g^{\lambda\nu} R_{\lambda 0 \nu 0} \approx \sum_{k=0}^{3} R_{k0k0} - R_{0000}$$

For a weak field, we may use the linear part of $R_{\lambda\mu\nu\rho}$:

$$R_{\lambda\mu\nu\rho} \approx \frac{1}{2} \left[ \frac{\partial^2 g_{\lambda\nu}}{\partial x^\rho \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\lambda} - \frac{\partial^2 g_{\lambda\rho}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\rho}}{\partial x^\nu \partial x^\lambda} \right]$$

For a static field, $R_{0000} \approx 0$

$$R_{i0j0} \approx \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j}$$

$$\rightarrow \quad G_{00} \approx 2 C_1 \left( \frac{1}{2} \eta^{ij} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} \right)$$

$$= C_1 \nabla^2 g_{00}.$$

Hence condition (4) implies $C_1 = 1$.

We have uniquely determined $\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$, known as the Einstein tensor.

The Einstein field equations are,

☆ $$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G_N T_{\mu\nu}}$$

# Comments on Einstein's Theory of Gravitation

Earlier in the course we developed a linear theory of gravitation using Einstein's equivalence principle as a guide.
In Einstein's geometric theory, general relativity, we were quietly guided by general coordinate invariance, namely that physical laws should hold in arbitrary coordinate systems. More precisely, the Principle of General Covariance states that a physical equation holds in a general gravitational field, if:

1) The eqn. holds in the absence of gravitation, i.e. it agrees w/ special relativity if $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^{\mu}_{\nu\lambda} = 0$.

and

2) The equation is generally covariant, i.e. it preserves its form under a general coordinate transformation $x \to x'$.

The Principle of General Covariance follows from the equivalence principle. Suppose we are in an arbitrary gravitational field, and suppose an equation satisfies (1) and (2) above. Consider a locally inertial coordinate system near any point. (Such a coordinate system exists by the equivalence principle.) Now condition (1) implies that the equation is valid in such a coordinate system, and (2) implies that it is then valid in any coordinate system. ■

## Number of Independent Components in $R^{\mu}{}_{\nu\lambda\sigma}$ (in 4D)

$R_{\mu\nu\lambda\sigma}$ is symmetric in exchange of $(\mu\nu)$ and $(\lambda\sigma)$, and antisymmetric in $\mu\nu$, and antisymmetric in $\lambda\sigma$.
There are $\frac{4\cdot 3}{2} = 6$ independent choices for $\mu\nu$, and 6 choices for $\lambda\sigma$.

→ There are $\frac{6\cdot 7}{2} = 21$ choices for $\mu\nu\lambda\sigma$.

The cyclic sum $\boxed{R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0}$, which is only <u>one</u> additional constraint because the cyclic sum is completely antisymmetric in exchange of its indices.

Finally, there are $21 - 1 = \boxed{20}$ independent components of the curvature tensor $R^{\mu}{}_{\nu\lambda\sigma}$.

The Ricci tensor $R_{\mu\nu}$ is symmetric in $\mu\nu$, so it has $\frac{4\cdot 5}{2} = \boxed{10}$ independent components.

This implies that the traceless part of $R^{\mu}{}_{\nu\lambda\sigma}$ has $20 - 10 = 10$ independent components. The traceless part is called the <u>Weyl tensor</u>, and with all lower indices has the form

$$\boxed{\begin{aligned} C_{\lambda\mu\nu\kappa} \equiv R_{\lambda\mu\nu\kappa} &- \tfrac{1}{2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) \\ &+ \tfrac{1}{6}R\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right) \end{aligned}}$$

☆ Note – Vacuum solutions to Einstein's equations can have nonvanishing Weyl tensor.

In 2D, $R_{\lambda\mu\nu\rho}$ has $\frac{1\cdot 2}{2} = 1$ independent component.
(1 choice for $(\lambda\mu)$ or $(\nu\rho)$, symmetric in $(\lambda\mu) \leftrightarrow (\nu\rho)$.

In this case $R_{\lambda\mu\nu\rho}$ can be written in terms of the curvature scalar $R$: $\boxed{R_{\lambda\mu\nu\rho} = \frac{1}{2} R \left( g_{\lambda\nu} g_{\mu\rho} - g_{\lambda\rho} g_{\mu\nu} \right)}$

– in 2D only!

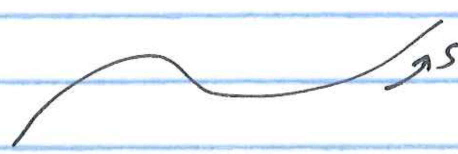
The Gaussian curvature $K$ is defined for a 2D manifold as $\boxed{K \equiv -R/2}$.

The Gaussian curvature is coordinate invariant. It is a description of the geometry intrinsic to the manifold.

In 1D, $R_{\lambda\mu\nu\rho} = 0$ by antisymmetry in $\lambda\mu$ or $\nu\rho$, and the fact that there is only one choice for $\lambda = \mu = \nu = \rho = 1$.

Hence all 1D manifolds are flat. The metric can always be chosen to be $g_{11} = \pm 1$ by a coordinate transformation,

$$g'_{11} = \left( \frac{dx}{dx'} \right)^2 g_{11}$$




The arc-length along the curve can be used as the coordinate, so $ds^2 = dx^2$.

## Coordinate Conditions

Weinberg 7.4 The Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is symmetric and has 10 independent components. However, the 10 components are related by the 4 Bianchi identities, $G^{\mu}{}_{\nu;\mu} = 0$.

This leaves effectively $10-4=6$ equations for the 10 independent components of $g_{\mu\nu}$.

The remaining 4 degrees of freedom are not fixed by the Einstein equations. This corresponds to the ability to transform any solution by an arbitrary coordinate transformation $x \to x'(x)$.

The nonlinear generalization of the harmonic gauge condition $\partial_\mu h^\mu{}_\nu = \frac{1}{2}\partial_\nu h^\mu{}_\mu$ in the weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$), is

$$\boxed{g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0}$$ Harmonic coordinate conditions.

To linear order in $h_{\mu\nu}$,

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} \simeq \eta^{\mu\nu}\cdot\frac{1}{2}\eta^{\lambda\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right)$$

$$= \partial_\mu h^{\lambda\mu} - \frac{1}{2}\partial^\lambda h^\mu{}_\mu$$

$= 0$ in harmonic coordinates.

Weinberg 7.5

## The Cauchy Problem

Initial conditions: Suppose $g_{\mu\nu}(t_0, \vec{x})$, $\left.\frac{\partial}{\partial x^0} g_{\mu\nu}\right|_{t_0, \vec{x}}$ are known.

If we knew $\frac{\partial^2 g_{\mu\nu}}{\partial (x^0)^2}$ from the Einstein equations

then we could integrate to find $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^0}$ at later times.

The spatial components $G_{ij} = -8\pi G_N T_{ij}$ can be used to solve for $\frac{\partial^2 g_{ij}}{\partial (x^0)^2}$.

However, $G_{\mu 0}$ contains no time derivatives higher than $\frac{\partial g_{\mu\nu}}{\partial x^0}$, as can be seen by the Bianchi identities

$$0 = D_\mu G^{\mu\nu} = +\frac{\partial}{\partial x^0} G^{\mu 0} + \frac{\partial}{\partial x^i} G^{\mu i} + \Gamma^\mu_{\nu\lambda} G^{\lambda\nu} + \Gamma^\nu_{\nu\lambda} G^{\mu\lambda}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^0} G^{\mu 0} \equiv \underbrace{-\frac{\partial}{\partial x^i} G^{\mu i} - \Gamma^\mu_{\nu\lambda} G^{\lambda\nu} - \Gamma^\nu_{\nu\lambda} G^{\mu\lambda}}_{\text{At most 2 time derivatives}}}$$

$\rightarrow G^{\mu 0}$ has at most 1 time derivative, as claimed.

The Einstein equations do not allow the Cauchy problem for $g_{\mu 0}$ to be solved. This is due to the coordinate independence.

The 4 equations $G_{\mu 0} = -8\pi G_N T_{\mu 0}$
must be imposed as constraints on initial data.

At time $x^0 = t_0$, suppose this eqn. is satisfied.
The Bianchi identity at $x^0 = t_0$ gives

$$\frac{\partial}{\partial x^0}\left(G^{\mu 0} + 8\pi G_N T^{\mu 0}\right)\Big|_{x^0 = t_0} = 0$$

This can be integrated to give $G^{\mu 0} = -8\pi G_N T^{\mu 0}$
at time $t_0 + \Delta t$, and then for all times $x^0$.