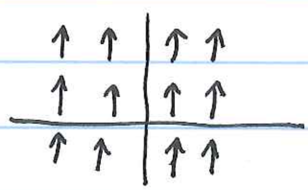
Weinberg 4.9
Zee V.6

Constant Vector Fields

In locally Cartesian coordinates ξ^α that map to the neighborhood of some point on a manifold, we can define a vector field to be constant in that neighborhood if $\frac{\partial V^\mu}{\partial \xi^\alpha} = 0$.



$V^x = 0$, $V^y = 1$ constant

We will use the notation V^μ_{LI} to represent the components of V^μ in the locally flat (inertial) Cartesian coordinates in the generically curved space(-time). In a general coordinate system the vector V^μ has components $V^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha} V^\alpha_{LI}$, so that $V^\mu_{LI} = \frac{\partial \xi^\mu}{\partial x^\alpha} V^\alpha$.

The condition that a vector field be constant becomes in general coordinates,

$$0 = \frac{\partial V^\mu_{LI}}{\partial \xi^\alpha} = \frac{\partial x^\nu}{\partial \xi^\alpha} \frac{\partial}{\partial x^\nu}\Big[\underbrace{V^\delta(x)\frac{\partial \xi^\mu}{\partial x^\delta}}_{V^\mu_{LI}}\Big]$$

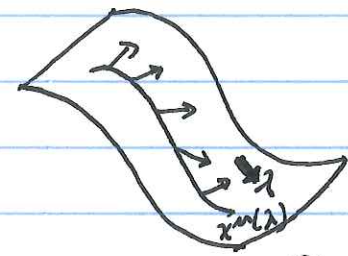
$$= \frac{\partial x^\nu}{\partial \xi^\alpha}\left[\frac{\partial \xi^\mu}{\partial x^\delta}\frac{\partial V^\delta}{\partial x^\nu} + V^\delta \frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\delta}\right]$$

$$= \frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial \xi^\mu}{\partial x^\delta}\Big[\frac{\partial V^\delta}{\partial x^\nu} + V^\sigma \underbrace{\frac{\partial x^\delta}{\partial \xi^\beta}\frac{\partial^2 \xi^\beta}{\partial x^\nu \partial x^\sigma}}_{\Gamma^\delta_{\nu\sigma}}\Big], \quad \text{using } \frac{\partial \xi^\mu}{\partial x^\delta}\frac{\partial x^\delta}{\partial \xi^\beta} = \delta^\mu_\beta$$

$$\Rightarrow \boxed{\frac{\partial V^\mu_{LI}}{\partial \xi^\alpha} = \frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial \xi^\mu}{\partial x^\delta} V^\delta_{;\nu}}$$

The condition for a vector field to be constant in curved space(-time) is $V^\mu{}_{;\alpha} = \boxed{D_\alpha V^\mu = 0.}$

Covariant Derivative Along a Curve



For a vector field defined along a curve (like $x^\mu(\tau)$ describing the trajectory of a particle), we can define a covariant derivative that transforms under coordinate transformations like a vector:

$$\boxed{\frac{DV^\sigma}{D\lambda} \equiv \frac{dV^\sigma}{d\lambda} + \Gamma^\sigma_{\mu\nu}\frac{dx^\nu}{d\lambda}V^\mu}$$

If V^μ is also defined in a neighborhood of the curve, then we can understand the covariant derivative this way:

$$\frac{DV^\mu_{LI}}{D\lambda} = \lim_{\Delta\to 0}\frac{V^\mu_{LI}(\Delta+\lambda) - V^\mu_{LI}(\lambda)}{\Delta}$$

$$= \frac{d\xi^\alpha}{d\lambda}\frac{\partial V^\mu_{LI}}{\partial\xi^\alpha} = \frac{d\xi^\alpha}{d\lambda}\frac{\partial x^\nu}{\partial\xi^\alpha}\frac{\partial\xi^\mu}{\partial x^\sigma}D_\nu V^\sigma$$

$$= \frac{\partial\xi^\mu}{\partial x^\sigma}\left(\frac{dx^\nu}{d\lambda}D_\nu V^\sigma\right) \equiv \frac{\partial\xi^\mu}{\partial x^\sigma}\frac{DV^\sigma}{D\lambda}$$

$$\frac{DV^\sigma}{D\lambda} = \frac{dx^\nu}{d\lambda}\left(\frac{\partial V^\sigma}{\partial x^\nu} + \Gamma^\sigma_{\mu\nu}V^\mu\right)$$

$$\rightarrow \frac{DV^\sigma}{D\lambda} = \frac{dV^\sigma}{d\lambda} + \Gamma^\sigma_{\mu\nu}\frac{dx^\nu}{d\lambda}V^\mu$$, as claimed.

Zee VI.1, IX.1

Parallel Transport of Vectors

Keep vector constant with respect to itself along a trajectory.

$\frac{DV^\mu}{D\lambda} = 0$ defines parallel transport

$$\boxed{\frac{dV^\mu}{d\lambda} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\lambda}V^\nu}$$ Parallel transport equation

In flat space, the components of V^μ remain constant in Cartesian coordinates:

Along a geodesic, the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$ is parallel transported.

$$\frac{D}{D\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0$$

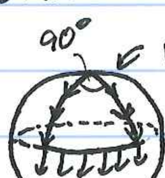
$$\rightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$

-Geodesic Equation

Weinberg Ch.6 Zee V.6

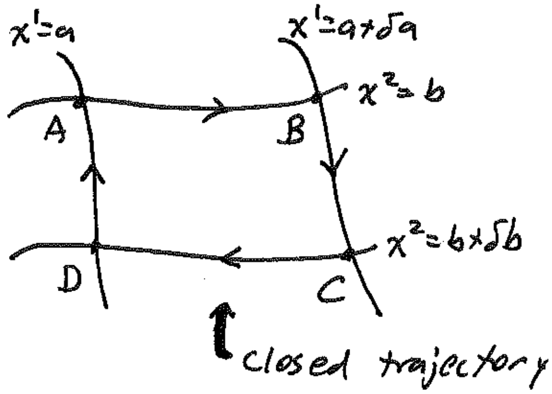
Curvature

In curved spaces, parallel transport of a vector along a closed loop does not generally return the vector to itself.



vector does not return to itself.

Definition: A manifold is flat if any vector parallel transported along any closed loop returns to itself.



Lines of constant coordinates

Along path from A to B: $\frac{DV^\alpha}{D\lambda}=0$ for vector field $V^\alpha(x)$ parallel transported.

$$\frac{dV^\alpha}{d\lambda} = -\Gamma^\alpha_{\mu 1}\frac{dx^1}{d\lambda}V^\mu \ , \ \frac{\partial V^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\nu}V^\nu = 0 \quad \text{along trajectory}$$

$$V^\alpha(B) = V^\alpha(A) + \int_{x^2=b} dx^1 \left(-\Gamma^\alpha_{\mu 1}V^\mu\right)$$

Similarly, $V^\alpha(C) = V^\alpha(B) + \int_{x^1=a+\delta a} dx^2\left(-\Gamma^\alpha_{\mu 2}V^\mu\right)$

$$V^\alpha(D) = V^\alpha(C) + \int_{x^2=b+\delta b} dx^1\left(\Gamma^\alpha_{\mu 1}V^\mu\right)$$

$$V^\alpha(A_{return}) = V^\alpha(D) + \int_{x^1=a} dx^2\left(\Gamma^\alpha_{\mu 2}V^\mu\right)$$

$$V^\alpha(A_{return}) - V^\alpha(A) = \int_{x^1=a} dx^2\, \Gamma^\alpha_{\mu 2}V^\mu - \int_{x^1=a+\delta a} dx^2\, \Gamma^\alpha_{\mu 2}V^\mu$$

$$+ \int_{x^2=b+\delta b} dx^1\, \Gamma^\alpha_{\mu 1}V^\mu - \int_{x^2=b} \Gamma^\alpha_{\mu 1}V^\mu dx^1$$

$$= \int_b^{b+\delta b} dx^2\, \delta a\left(-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 2}V^\mu\right)\right) + \int_a^{a+\delta a} dx^1\, \delta b\left(\frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\mu 1}V^\mu\right)\right)$$

$$= \delta a\, \delta b\left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 2}V^\mu\right) + \frac{\partial}{\partial x^2}\left(\Gamma^\alpha_{\mu 1}V^\mu\right)\right]$$

$V^\alpha(A_{return}) - V^\alpha(A)$

$$= \delta a\,\delta b \left\{ -\left(\frac{\partial}{\partial x^1}\Gamma^\alpha_{\mu 2}\right)V^\mu - \Gamma^\alpha_{\mu 2}\frac{\partial V^\mu}{\partial x^1} + \left(\frac{\partial}{\partial x^2}\Gamma^\alpha_{\mu 1}\right)V^\mu + \Gamma^\alpha_{\mu 1}\frac{\partial V^\mu}{\partial x^2} \right\}$$

The vector V^μ is parallel transported along the loop, so

$$\frac{\partial V^\beta}{\partial x^1} = -\Gamma^\beta_{\mu 1} V^\mu \quad \text{or} \quad \frac{\partial V^\beta}{\partial x^2} = -\Gamma^\beta_{\mu 2} V^\mu$$

along appropriate portions of the loop.

$$\Rightarrow \delta V^\alpha \equiv V^\alpha(A_{return}) - V^\alpha(A)$$

$$= \delta a\,\delta b \left\{ \frac{\partial}{\partial x^2}\Gamma^\alpha_{\mu 1} - \frac{\partial}{\partial x^1}\Gamma^\alpha_{\mu 2} + \Gamma^\alpha_{\beta 2}\Gamma^\beta_{\mu 1} - \Gamma^\alpha_{\beta 1}\Gamma^\beta_{\mu 2} \right\} V^\mu$$

$$\equiv \delta a\,\delta b\, R^\alpha_{\mu 1 2} V^\mu$$

2 ← δb in x^2-direction

1 ← δa in x^1-direction

More generally, if V^μ is parallel transported around a loop spanning δa in x^σ-direction, δb in x^λ-direction $(\sigma \neq \lambda)$:

$$\delta V^\alpha = \delta a\,\delta b\, R^\alpha_{\mu\sigma\lambda} V^\mu \quad \text{, where}$$

$$\boxed{R^\alpha_{\mu\sigma\lambda} \equiv \frac{\partial \Gamma^\alpha_{\mu\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\alpha_{\mu\lambda}}{\partial x^\sigma} + \Gamma^\alpha_{\delta\lambda}\Gamma^\delta_{\mu\sigma} - \Gamma^\alpha_{\delta\sigma}\Gamma^\delta_{\lambda\mu}}$$

<u>Riemann Curvature tensor</u>

<u>Tedious Exercise</u>: Show that $R^\alpha_{\mu\sigma\lambda}$ is a tensor.

☆ Note: Space(time) is flat iff $R^\alpha_{\mu\sigma\lambda} = 0$ everywhere.

Properties of $R^\alpha{}_{\mu\sigma\lambda}$:

1) $R^\alpha{}_{\mu\sigma\lambda}$ is the only tensor that can be constructed from $g_{\mu\nu}$ and its first and second derivatives.

2) $R^\alpha{}_{\mu\sigma\lambda}$ can also be defined in terms of the commutator of covariant derivatives:

$$\boxed{V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -V_\sigma R^\sigma{}_{\mu\nu\kappa}}$$

$$\boxed{V^\lambda{}_{;\nu;\kappa} - V^\lambda{}_{;\kappa;\nu} = V^\sigma R^\lambda{}_{\sigma\nu\kappa}}$$

3) Define $R_{\lambda\mu\nu\kappa} \equiv g_{\lambda\sigma} R^\sigma{}_{\mu\nu\kappa}$

$$\boxed{R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left\{\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right\} + g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right]}$$

4)
$$\left.\begin{aligned}
&R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \\
&R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu} \\
&R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0
\end{aligned}\right\}\ \text{Algebraic Relations}$$

Useful contractions of $R^{\lambda}{}_{\mu\nu\kappa}$:

$R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$ Ricci tensor

$R \equiv g^{\mu\kappa} R_{\mu\kappa}$ Curvature Scalar

<u>Bianchi Identities</u>

In a locally Cartesian (inertial) coordinate system,
$\Gamma^{\lambda}_{\mu\nu} = 0$, but $\frac{\partial}{\partial x^{\alpha}} \Gamma^{\lambda}_{\mu\nu} \neq 0$.

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2} \frac{\partial}{\partial x^{\eta}} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa} \partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu} \partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \right)$$

<u>Exercise</u>: $\boxed{R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0}$ cyclic permutations

This is a covariant relation so it is true in arbitrary frames.

Contract with $g^{\lambda\nu}$:

$$\boxed{R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0}$$

Contract w/ $g^{\mu\kappa}$:

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0$$

$$\Rightarrow \left(R^{\mu}{}_{\eta} - \frac{1}{2} \delta^{\mu}{}_{\eta} R\right)_{;\mu} = 0$$

$$\boxed{\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0}$$