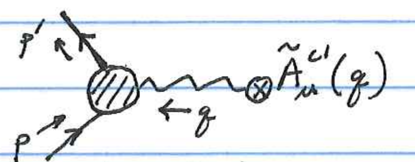
Electron Vertex Function

Consider scattering of an electron off of a background electromagnetic field. The corresponding Feynman diagrams have the form:



$$\equiv -i\bar{u}(p')\,\tilde{\Gamma}^\mu(p,p')\,u(p)\,\tilde{A}^{cl}_\mu(p'-p)$$

$(q = p'-p)$

$$\tilde{A}^{cl}_\mu(q) = \int d^4x\, e^{iq\cdot x} A^{cl}_\mu(x)$$

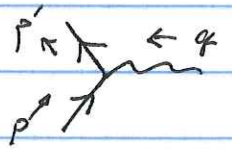
You can think of the background electromagnetic field as due to an external current j_μ.

In Lorenz gauge, $\Box A^{cl}_\mu = e j_\mu$, i.e.

$$\boxed{\tilde{A}^{cl}_\mu(q) = -\frac{e}{q^2}\,\tilde{j}_\mu(q)}$$

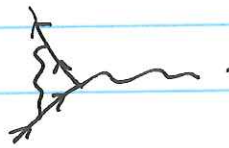
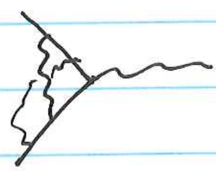
The 4×4 matrix function of p, p' $\Gamma^\mu(p,p')$ is called the electron vertex function.

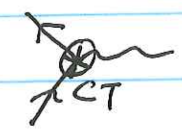
At tree level,



$$\boxed{-i\,\Gamma^{(1)\mu}(p,p') = -ie\gamma^\mu}$$

Higher order corrections to the vertex function include

$-i\tilde{\Gamma}^\mu(p,p') =$  $+$  $+$  $+ \dots$

$+$ 

The counterterm comes from $\mathcal{L}_{CT} \supset -D\bar{\Psi} e \not{A} \Psi$,

$$\boxed{-i\tilde{\Gamma}^\mu(p,p') \supset -iDe\gamma^\mu}$$

Higher order corrections will not all be proportional to γ^μ. Lorentz invariance and gauge invariance restrict the possible matrix structures appearing in $\tilde{\Gamma}^\mu$. We can decompose $\tilde{\Gamma}^\mu$ as:

$$\tilde{\Gamma}^\mu(p,p') = A\gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu)$$

When $\tilde{\Gamma}^\mu(p,p')$ appears in an S-matrix element it is sandwiched between spinor wavefunctions $\bar{u}(p')$ and $u(p)$. The on-shell momenta and wavefunctions satisfy $p^2 = p'^2 = m^2$, $\not{p}\, u(p) = m\, u(p)$
$\bar{u}(p')\not{p}' = m\,\bar{u}(p')$

We can think of A, B, C as functions of $(p'-p)^2$, as any other Lorentz invariants are redundant, ie

$A = A(q^2)$, $B = B(q^2)$, $C = C(q^2)$, $q^\mu = p'^\mu - p^\mu$.

(Note that we didn't include matrix structures including γ^5 because they would violate parity.)

Another Ward Identity due to gauge invariance implies that $\boxed{\text{inside S matrix elements, } q_\mu \tilde{\Gamma}^\mu = 0}$

$$0 = q_\mu \, \bar{u}(p') \tilde{\Gamma}^\mu(p,p') u(p)$$

$$= A(q^2)\, \bar{u}(p') (\not{p}' - \not{p}) u(p) + B(q^2)\, \bar{u}(p') u(p) (p'^\mu - p^\mu)(p'_\mu + p_\mu)$$
$$+ C(q^2)\, q^2$$

Since $\bar{u}(p')\not{p} u(p) = \bar{u}(p') \not{p}' u(p)$, the term $\propto A(q^2)$ vanishes. Since $p^2 = p'^2 = m^2$, the term proportional to $B(q^2)$ vanishes.

If the photon is not on-shell, $q^2 \neq 0$. Hence, the Ward Identity implies $C(q^2) = 0$. Hence $\tilde{\Gamma}^\mu$ takes the form,

$$\boxed{\tilde{\Gamma}^\mu(p,p') = A(q^2)\gamma^\mu + B(q^2)(p'^\mu + p^\mu)}$$

It is customary to rewrite the second term using an identity for solutions to the Dirac equation $u(p)$, $u(p')$:

$$\boxed{\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m} \bar{u}(p') \left(p'^\mu + p^\mu + i\sigma^{\mu\nu} q_\nu \right) u(p)}$$

Gordon identity

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

The Gordon identity follows from

$$\gamma^\mu \not{p} = \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}]$$
$$= p^\mu - i\sigma^{\mu\nu} p_\nu$$

and $\not{p}'\gamma^\mu = p'^\mu + i\sigma^{\mu\nu} p'_\nu$. Then,

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu \not{p})u$$

$$= \frac{1}{2m}\bar{u}(p')(p'^\mu + p^\mu + i\sigma^{\mu\nu} q_\nu)u(p) \text{ as claimed.}$$

The part $\propto p'^\mu + p^\mu$ has the form of the term in $\bar{u}\tilde{\Gamma}^\mu u$ multiplying $B(q^2)$. Hence, instead of using γ^μ and $(p'^\mu + p^\mu)$ in the decomposition of $\tilde{\Gamma}^\mu$, we can use γ^μ and $i\sigma^{\mu\nu}q_\nu$:

$$\boxed{\tilde{\Gamma}^\mu(p,p') = e\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} e F_2(q^2)}$$

Our goal is to interpret the <u>form factors</u> $F_1(q^2)$ and $F_2(q^2)$ and calculate them at 1-loop.

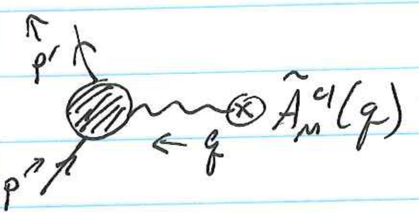
$F_1(q^2) \equiv$ Dirac Form Factor
$F_2(q^2) \equiv$ Pauli Form Factor.

At tree level, $F_1(q^2) = 1$, $F_2(q^2) = 0$.

To understand the physical meaning of the form factors we consider scattering of a nonrelativistic electron from a background electromagnetic field.

First, consider a background electrostatic potential $\phi(\vec{x})$, s.t. $A_\mu^{cl}(x) = (\phi(\vec{x}), \vec{0})$.

$$\tilde{A}_\mu^{cl}(q) = \int d^4x\, e^{iq\cdot x} A_\mu^{cl}(x) = \left(2\pi\delta(q^0)\tilde{\phi}(\vec{q}), \vec{0}\right)$$



$$i\mathcal{M} = -i\bar{u}(p')\tilde{\Gamma}^\mu(p,p')u(p)\tilde{A}_\mu^{cl}(p'-p)$$
$$= -i\bar{u}(p')\tilde{\Gamma}^0(p,p')u(p)\tilde{\phi}(\vec{p}'-\vec{p})$$
(with the $2\pi\delta(q^0)$ factored out)

If the electrostatic field varies slowly over a large region, then $\tilde{\phi}(\vec{q})$ will be concentrated at $\vec{q} \simeq \vec{0}$.

In that limit, $\tilde{\Gamma}^0(p,p') = e\gamma^0 F_1(q^2) + \frac{i\sigma^{0j}}{2m} e q_j F_2(q^2)$

$$\rightarrow e\gamma^0 F_1(q^2)$$

In the nonrelativistic limit $\bar{u}(p')\gamma^0 u(p) = 2m\,\xi'^\dagger\xi$.
The scattering amplitude becomes,

$$\boxed{i\mathcal{M} \approx -ieF_1(0)\,\tilde{\phi}(\vec{q})\cdot 2m\,\xi'^\dagger\xi}$$

This agrees with the Born approximation for scattering off of a potential $\boxed{V(\vec{x}) = e F_1(0)\, \phi(\vec{x}).}$

$\boxed{\text{Hence, } eF_1(0) \text{ is the electric charge of the electron.}}$

Since $F_1(0) = 1$ at tree level, radiative corrections to $F_1(q^2)$ should vanish at $q^2 = 0$. $\rightarrow$ <u>Renormalization Condition</u>

Now consider scattering off of a magnetic field specified by a static vector potential $A_\mu^{cl}(x) = (0, \vec{A}^{cl}(\vec{x}))$.

Then the scattering amplitude is,

$$iM = \sum_j ie\left[\bar{u}(p')\left(\gamma^j F_1(q^2) + \frac{i\sigma^{j\nu}q_\nu}{2m} F_2(q^2)\right)u(p)\right] \tilde{A}^j_{cl}(\vec{q})$$

Again we assume that $\tilde{A}^j_{cl}(\vec{q})$ is peaked near $\vec{q} = \vec{0}$.

Consider the spinors in the Weyl basis, expanded in $\vec{p}$ and $\vec{p}\,'$:

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \\ \left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

Consider the $F_1(q^2)$ term in iM:

$$iM \supset ie\,\bar{u}(p')\gamma^j F_1(q^2)\, u(p)\, \tilde{A}^j_{cl}(\vec{q})$$

<u>Exercise:</u>

$$\approx ie\cdot 2m\, \xi'^\dagger\left(\frac{\vec{p}\,'\cdot\vec{\sigma}}{2m}\sigma^j + \sigma^j\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi\, F_1(0)\tilde{A}^j_{cl}(\vec{q})$$

Use $\sigma^i\sigma^j = \delta^{ij} + i\,\epsilon^{ijk}\sigma^k$ (summed over repeated indices)

$$iM \approx ie\,\frac{\cancel{2m}}{\cancel{2m}}\,\xi'^{\dagger}(\vec{p}\,'+\vec{p})\cdot\vec{\tilde{A}}_{cl}(\vec{q})\,\xi\,F_1(0)$$

$$+ie\,\frac{\cancel{2m}}{\cancel{2m}}\,\xi'^{\dagger}(+i\epsilon^{ijk}q^j\sigma^k)\,\xi\,\tilde{A}^i_{cl}(\vec{q})\,F_1(0)$$

The first term, w/ $\vec{p}\,'\approx\vec{p}$, corresponds to the $(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p})$ term in the nonrelativistic Hamiltonian.

The second term contributes to the electron's magnetic moment interaction:
The magnetic field is $\vec{B} = \nabla\times\vec{A}_{cl}$. Fourier transform:

$$\tilde{B}^k(\vec{q}) = \int d^3x\, e^{+i\vec{q}\cdot\vec{x}}\,\epsilon_{kij}\,\partial_i A^{cl}_j(\vec{x})$$ (Euclidean conventions — repeated indices summed, no minus signs)

$$= -iq^i\epsilon^{kij}\int d^3x\, e^{+i\vec{q}\cdot\vec{x}}A^j_{cl}(\vec{x})$$ (integration by parts)

$$= -i\,\epsilon^{kij}q^i\tilde{A}^j_{cl}(\vec{q})$$

Then the second term in iM above is:

$$iM \approx -ie\,\xi'^{\dagger}\sigma^k\xi\,\tilde{B}^k(\vec{q})\;F_1(0)$$

This is the Born approximation to scattering off a potential,

$$V(\vec{x}) = -\vec{\mu}_1\cdot\vec{B}\quad,\quad \boxed{\vec{\mu}_1 = \frac{e}{m}F_1(0)\,\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi}$$

The $F_2(q^2)$ term in iM contributes similarly:

$$iM \supset +\frac{e}{2m} \cdot \bar{u}(p') \frac{i}{2}[\gamma^j, \gamma^k] q^k F_2(q^2) u(p) \tilde{A}^j_{cl}(\vec{q})$$

Use again $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$$\rightarrow [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

Then $\bar{u}(p')\sigma^{jk} q^k u(p) \approx 2m\, \xi'^{\dagger} \epsilon^{jki} q^k \sigma^i \xi$

$$iM \gtrapprox \frac{e}{\cancel{2m}} \cdot \cancel{2m}\, \xi'^{\dagger}\sigma^i\xi\, \epsilon^{jki} q^k F_2(q^2) \tilde{A}^j_{cl}(\vec{q})$$

$$= -ie\, \xi'^{\dagger}\sigma^i\xi\, \tilde{B}^i(\vec{q})\, F_2(q^2)$$

Comparing again w/ the Born approximation, we get another contribution to the magnetic moment,

$$\vec{\mu}_2 = \frac{e}{m} F_2(0)\, \xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi$$

where we set $q^2 = 0$ in $F_2(q^2)$ because we assume $\tilde{A}^j_{cl}(\vec{q})$ is dominated by $\vec{q} \approx \vec{0}$.

Adding everything together, we find a magnetic moment interaction,

$$\boxed{\begin{gathered} V(\vec{x}) = -\vec{\mu} \cdot \vec{B}(\vec{x}) \\ \vec{\mu} = \frac{e}{m}\left[F_1(0) + F_2(0)\right] \xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi \end{gathered}}$$

$\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi$ is the electron spin $\vec{S}$. It is common to write the electron magnetic moment in the form,

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}, \quad g \equiv \text{Landé } g\text{-factor.}$$

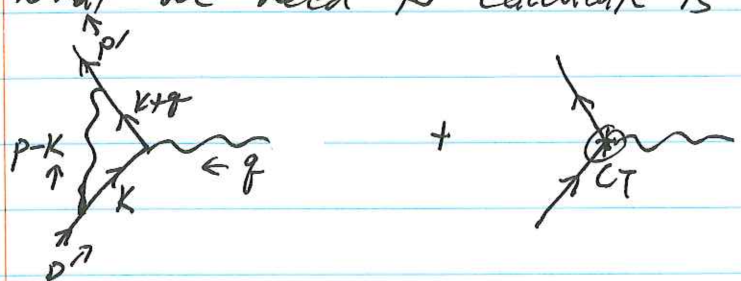
Since we have argued that $F_1(0) = 1$, we have derived,

$$\boxed{g = 2 + 2F_2(0)}$$

$g=2$ was the prediction of the Dirac equation. At tree level $F_2(0)=0$, so $F_2(0) = \mathcal{O}(\alpha)$.

The value of $(g-2)$ is called the <u>anomalous magnetic moment of the electron</u>.

What we need to calculate is



The counterterm is chosen so that the electron charge is e, so we expect the same counterterm which set $\frac{d\Sigma}{d\not{p}}\big|_{\not{p}=m}=0$ for the electron self energy to also set $F_1(0)=1$. This is a consequence of a Ward Identity, but we will have to wait to see this.