We are now in a position to construct a non-Abelian gauge theory from knowledge of the gauge group and the representations under which the matter fields transform.

Example: Quantum Chromodynamics = The Strong Interaction

- SU(3) gauge theory
- Six flavors of quarks (u,d,c,s,t,b), each of which transforms in the fundamental representation of the SU(3) gauge group.

$SU(3) \Rightarrow 8$ gauge fields $A_\mu^a,\ a=1,\dots,8$
(because $\dim SU(3)=8$)

Fundamental rep = defining rep = 3-dimensional for SU(3)
$\Rightarrow$ 3 colors of quarks for each flavor.

Label quark fields $\underline{q_I^i}$, $I=u,d,c,s,t,b$ — flavor
$i=1,2,3$ — color

Generators of SU(3) $\underline{T^a}$, $a=1,\dots,8$

Gauge coupling $\underline{g_3}$

The Lagrangian for QCD is then:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \sum_{I,i,j} \bar{q}_I^i \left(i\partial\!\!\!/\,\delta_{ij} - g_3 A\!\!\!/^a T^a_{ij} \right) q_I^j$$

+ quark mass terms.

In the Standard Model quark (and lepton) masses are due to couplings to the Higgs fields, which requires one more ingredient...

Pseudoreality of SU(2) Representations:

A representation of a group is called pseudoreal if there is a nonsingular matrix S such that

$$\boxed{S T^a S^{-1} = -T^{a*}}.$$

In that case a representation is equivalent to its conjugate.

The $SO(N)$ groups have all real representations. Since $SU(2) \sim SO(3)$, the integer-spin reps of $SU(2)$ are real.
The half-odd-integer spin reps are called spinor reps. For them, there is a nontrivial S that relates the rep to its conjugate.

Consider the spin-$1/2$ representation:

$$T^1 = \frac{\sigma^1}{2}, \quad T^2 = \frac{\sigma^2}{2}, \quad T^3 = \frac{\sigma^3}{2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^{1*} \Rightarrow \boxed{-T^{1*} = -T^1}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\sigma^{2*} \Rightarrow \boxed{-T^{2*} = +T^2}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{3*} \Rightarrow \boxed{-T^{3*} = -T^3}$$

Let $\boxed{S = -i\sigma^2}$.

$$ST^1S^{-1} = \sigma^2 \frac{\sigma^1}{2} \sigma^2 = -(\sigma^2)^2 \frac{\sigma^1}{2} = -\frac{\sigma^1}{2} = -T_1^*$$

$$ST^2S^{-1} = \sigma^2 \frac{\sigma^2}{2} \sigma^2 = \frac{\sigma^2}{2} = -T_2^*$$

$$ST^3S^{-1} = \sigma^2 \frac{\sigma^3}{2} \sigma^2 = -\frac{\sigma^3}{2} = -T_3^*$$

Hence, $\boxed{ST^aS^{-1} = -T^{a*}}$ for $a=1,2,3$.

Suppose $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transforms as an $SU(2)$ doublet,

$$\Phi \to \exp\left[i\theta^a \frac{\sigma^a}{2}\right] \Phi$$

Then,

$$\Phi^* \to \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \Phi^*$$

The combination $\boxed{i\sigma^2 \Phi^* = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}}$ transforms as an $SU(2)$ doublet, as we will now check.

$$i\sigma^2 \Phi^* \to i\sigma^2 \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \Phi^*$$
$$= i\sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n \Phi^*$$

Expand $\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n$ and insert $(-i\sigma^2)(i\sigma^2) = 1$ between factors of $(\sigma^a)^T$, and to the right of the last factor.

Then, $i\sigma^2 \Phi^* \to \sum_n \frac{1}{n!}\left[i\theta^a \frac{\sigma^a}{2}\right]^n i\sigma^2 \Phi^*$

$$= \exp\left[i\theta^a \frac{\sigma^a}{2}\right]\left(i\sigma^2 \Phi^*\right)$$

This is what we wanted to prove:
Φ and $(i\sigma^2 \Phi^*)$ transform the same way.

The Higgs sector of the Standard Model takes advantage of this fact.

The Higgs fields $\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ are a pair of complex scalar fields that transform as an $SU(2)_w$ doublet under the weak interactions. — (This is a gauge invariance, not a global symmetry.)

The left-handed fermions (quarks or leptons) form $SU(2)_w$ doublets.

The right-handed fermions are $SU(2)_w$ singlets, i.e. they are invariant under $SU(2)_w$ transformations.

The Lagrangian $\boxed{\mathcal{L}_{Yuk_d} = \lambda_d \left(\bar{\Psi}_L^1, \bar{\Psi}_L^2\right)\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \psi_R}$ + h.c.

is $SU(2)_w$-invariant. But so is

$$\boxed{\mathcal{L}_{Yuk_u} = \lambda_u \left(\bar{\Psi}_L^1, \bar{\Psi}_L^2\right) i\sigma^2 \begin{pmatrix}\phi_1^* \\ \phi_2^*\end{pmatrix} \psi_R'} + \text{h.c.}$$

Both types of Yukawa couplings are required in the Standard Model for understanding fermion masses.

If the Higgs doublet has a potential $V(\phi_1, \phi_2)$ with a minimum at $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}0\\ v\end{pmatrix}$ with v real,

then expanding the Lagrangian about this minimum gives rise to terms of the form

$$\mathcal{L}_d = \lambda_d (\bar{\psi}_L^1, \bar{\psi}_L^2)\begin{pmatrix}0\\ v\end{pmatrix}\psi_R \quad + h.c.$$

$$= (\lambda_d v)\bar{\psi}_L^2 \psi_R + h.c.$$

This looks like a mass term for the Dirac spinor whose left-handed component is ψ_L^2 and whose right-handed component is ψ_R, with mass $(\lambda_d v)$.

$$\mathcal{L}_u = \lambda_u (\bar{\psi}_L^1, \bar{\psi}_L^2)\begin{pmatrix}v\\ 0\end{pmatrix}\psi_R' \quad + h.c.$$

$$= (\lambda_u v)\ \bar{\psi}_L^1 \psi_R' + h.c.$$

This is a mass term for ψ_L^1 and ψ_R'.

If it weren't for pseudoreality of $SU(2)$ representations, we couldn't have written $\mathcal{L}_u$ consistent w/ the weak scalar, and ψ' would be massless.

Spontaneous Symmetry Breaking (SSB)

(Cheng + Li ch. 5.3)

Normally, symmetries lead to degeneracy between states that form <u>multiplets</u> of the symmetry group. The states in a multiplet transform in irreducible representations of the symmetry group.

Suppose $U = \exp(i\theta^a Q^a)$ is an element of the symmetry group, i.e. $U \underset{\uparrow \text{Hamiltonian}}{H} U^\dagger = H$.

Suppose $H|A\rangle = E_A |A\rangle$.
Define $|B\rangle \equiv U|A\rangle$.
Then $H|B\rangle = UHU^\dagger|B\rangle = UH|A\rangle = E_A U|A\rangle = E_A|B\rangle$.
Hence, $|B\rangle$ is an eigenstate of H w/ eigenvalue E_A.

Suppose ϕ_A and ϕ_B are two fields related by the symmetry transformation, $U\phi_A U^\dagger = \phi_B$.

If $|A\rangle = \phi_A \underset{\text{vacuum}}{|0\rangle}$

$$|B\rangle = \phi_B|0\rangle = U\phi_A U^\dagger|0\rangle$$
$$= U|A\rangle \quad \underline{\text{if}} \quad U^\dagger|0\rangle = |0\rangle$$

However, if $U^\dagger|0\rangle \neq |0\rangle$, the states $|A\rangle$ and $|B\rangle$ defined above are generally nondegenerate.
In this case the vacuum is not invariant under the symmetry transformation, and we say that the symmetry is <u>spontaneously broken</u>.

Example: Ferromagnetism near the Curie temperature T_c

For $T > T_c$ spins are randomly aligned
$\rightarrow$ magnetization vanishes
For $T < T_c$ spins align
$\rightarrow$ $\exists$ net magnetization

Landau-Ginzburg model: Near T_c, free energy is a rotation-invariant function of the magnetization $\vec{M}$.

$$U(\vec{M}) = (\partial_i \vec{M}) \cdot (\partial_i \vec{M}) + V(\vec{M})$$
$$V(\vec{M}) = \alpha_1(T)\, \vec{M}\cdot\vec{M} + \alpha_2\, (\vec{M}\cdot\vec{M})^2 \quad , \alpha_2 > 0$$

Ground state: $\frac{\partial V}{\partial M_i} = 0 \Rightarrow \vec{M}\left(\alpha_1 + 2\alpha_2\, \vec{M}\cdot\vec{M}\right) = 0$

$T > T_c$: $\alpha_1(T) > 0$, min at $\vec{M} = 0$.
$T < T_c$: $\alpha_1(T) < 0$, min at $|\vec{M}| = \left(\frac{-\alpha_1}{2\alpha_2}\right)^{1/2}$

For $T < T_c$ $|\vec{M}| \neq 0$, but the direction is random
$\rightarrow$ decides which ground state

Symmetry breaking = noninvariance of vacuum
$\rightarrow$ nonvanishing order parameter.

Goldstone's Theorem

Given a continuous symmetry, Noether's theorem implies $\exists$ current $J^\mu(x)$ s.t. $\partial_\mu J^\mu = 0$.

If $\int d^3x\, J^0(x) \equiv Q$ is well defined, then it is conserved:

$$\boxed{0} = \int d^3x\, \partial_\mu J^\mu = \int d^3x\, \partial_0 J^0 + \int d^3x\, \nabla\cdot\vec{J}$$
$$\boxed{= \frac{d}{dt} Q} + \underbrace{\int d\vec{S}\cdot\vec{J}}_{=0 \text{ if } \vec{J} \text{ falls quickly enough @ } \infty}$$

Charges generate symmetry transformations: $U = \exp(i\theta^a Q^a)$.

Example: Momentum generates translations

$$\phi(x) = \exp(iP\cdot x)\,\phi(0)\,\exp(-iP\cdot x)$$

If the symmetry U is spontaneously broken, then $U|0\rangle \neq |0\rangle$.

Infinitesimal form: $(1 + i\theta^a Q^a)|0\rangle \neq |0\rangle$

$$\rightarrow \boxed{Q^a|0\rangle \neq 0}$$

Then
$$\langle 0|Q^2(t)|0\rangle = \int d^3x\, \langle 0|J_0(\vec{x},t)\,Q(t)|0\rangle$$
$$= \int d^3x\, \langle 0|e^{iP\cdot x} J_0(0) e^{-iP\cdot x}\, e^{iP\cdot x} Q(0) e^{-iP\cdot x}|0\rangle$$
$$= \int d^3x\, \langle 0|J_0(0)\,Q(0)|0\rangle$$
$= \infty$ since the integrand is indep. of $\vec{x}$.

However, commutators w/ Q are well defined.

Under an infinitessimal symmetry transformation,

$$\phi(x) \to \exp(i\theta Q)\,\phi(x)\,\exp(-i\theta Q)$$
$$\approx \phi(x) + i\theta\,[Q, \phi(x)]$$

If $\exp(-i\theta Q)|0\rangle = |0\rangle$, i.e. the symmetry is unbroken, then $\langle 0|[Q,\phi(x)]|0\rangle = 0$, and under a symmetry transformation $\langle 0|\phi(x)|0\rangle \to \langle 0|\phi(x)|0\rangle$

A well-defined definition of spontaneous symmetry breaking is the condition $\boxed{\langle 0|[Q(t), \phi(0)]|0\rangle \neq 0,}$ for some field ϕ.

This quantity is time independent:

$$0 = \int d^3x\,[\partial_\mu J^\mu(x), \phi(0)]$$
$$= \frac{d}{dt}\int d^3x\,[J^0(x), \phi(0)] + \underbrace{\int d\vec{S}\cdot[\vec{J}(x), \phi(0)]}_{\to 0}$$
$$= \frac{d}{dt}\,[Q(t), \phi(0)].$$

Hence, the condition for SSB is $\langle 0|[Q(t), \phi(0)]|0\rangle \equiv \eta \neq 0$

Insert a complete set of states s.t. $\sum_n |n\rangle\langle n| = 1$.

$$\eta = \sum_n \int d^3x\,\Big(\langle 0|J^0(x)|n\rangle\langle n|\phi(0)|0\rangle - \langle 0|\phi(0)|n\rangle\langle n|J^0(x)|0\rangle\Big)$$

$$= \sum_n \int d^3x\,\Big(e^{-ip_n\cdot x}\langle 0|J^0(0)|n\rangle\langle n|\phi(0)|0\rangle - e^{ip_n\cdot x}\langle 0|\phi(0)|n\rangle\langle n|J^0(0)|0\rangle\Big)$$

$$= \sum_n (2\pi)^3\,\delta^3(\vec{p}_n)\Big(e^{-i\omega_n t}\langle 0|J^0(0)|n\rangle\langle n|\phi(0)|0\rangle - e^{i\omega_n t}\langle 0|\phi(0)|n\rangle\langle n|J^0(0)|0\rangle\Big)$$

The terms in the sum w/ $\omega_n \neq 0$ would not be time independent, so they must vanish.
The term(s) with $\omega_n = 0$, $\vec{p}_n = 0$ must contribute to the sum because $\eta \neq 0$ by assumption.

Hence, $\exists$ a state $|n\rangle$ with $\omega_n = 0$, $\vec{p}_n = 0$.
If the one-particle states satisfy the relativistic dispersion relation $\omega_n^2 = \vec{p}_n^{\,2} + m_n^2$, then $|n\rangle$ is massless, $m_n = 0$.

This is <u>Goldstone's theorem</u>!
$\forall$ spontaneously broken continuous global symmetry, $\exists$ a massless <u>Goldstone boson</u> in the spectrum.