Renormalization

By studying the spectrum of free-field Hamiltonians, the mass of the particles created/annihilated by the free field was identified with a parameter in the Lagrangian.

In QED the coupling e was identified with the electric charge of the electron.

Calculation of tree-level cross sections confirmed the interpretation of the parameters in the Lagrangian. However, there is no reason to expect the simple relationship between Lagrangian parameters and physical observables to persist at higher order in perturbation theory. In fact, there is reason to expect the contrary.

An electron acts as a source of the electromagnetic field. In order to move an electron, you have to carry the electromagnetic field along with it. Even classically, we would expect the energy contained in the electromagnetic field to contribute to the electron's mass. If the electron were a charged shell with charge e, radius r, then the measured mass of the electron should be

$$\underbrace{m}_{\text{measured mass}} = \underbrace{m_0}_{\text{"bare mass"}} + \underbrace{\frac{e^2}{2rc^2}}_{\text{effect of interactions.}}$$

The <u>bare mass</u> m_0 is unphysical, but it would be the mass of the charged shell in the absence of the electromagnetic field.

It would be desirable for perturbation theory to be organized in terms of physical quantities. That is the goal of <u>renormalized perturbation theory</u>.

We will split the Lagrangian into three parts: <u>free</u>, <u>physical interactions</u>, and <u>counterterms</u>.

For example, the bare mass^2 appearing in the Lagrangian we will call m_0^2, and write

$$\boxed{\underset{\text{bare mass}}{m_0^2} = \underset{\text{physical mass}}{m^2} + \underset{\text{counterterm.}}{\delta m^2}}$$

The physical mass^2 will be considered as part of the free Lagrangian, but δm^2 will be considered part of the interactions. Then the mass of particles created and annihilated by the free field is the physical mass m.

Similarly, we will find that the electric charge can be screened due to interactions, like in a dielectric, so we call the coupling in the Lagrangian e_0, and write

$$\boxed{\underset{\text{bare charge}}{e_0} = \underset{\text{physical charge}}{e} + \underset{\text{counterterm.}}{\delta e}}$$

As our warm-up example, consider a theory of a complex scalar field ψ coupled to a real scalar field ϕ, with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu_0^2}{2}\phi^2 + |\partial_\mu \psi|^2 - m_0^2|\psi|^2 - g_0\, \psi^* \psi \phi$$

$$= \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{\mu^2}{2}\tilde{\phi}^2 + |\partial_\mu \tilde{\psi}|^2 - m^2|\tilde{\psi}|^2 - g\, \tilde{\psi}^* \tilde{\psi} \tilde{\phi} + \mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = A\tilde{\phi} + \frac{B}{2}(\partial_\mu \tilde{\phi})^2 - \frac{C}{2}\tilde{\phi}^2 + D|\partial_\mu \tilde{\psi}|^2 - E|\tilde{\psi}|^2 - F\,\tilde{\psi}^*\tilde{\psi}\tilde{\phi} + \text{const.}$$

We have introduced six counterterms $A, \ldots, F$ which should be determined by physical requirements. For example,

① "Meson" mass $= \mu$ $\qquad (\phi \equiv \text{Meson})$
② "Nucleon" mass $= m$ $\qquad (\psi \equiv \text{Nucleon})$
③ Measured coupling $= g$

We need three more conditions. A convenient choice will be:

④ $\langle 0|\tilde{\phi}|0\rangle = 0$
⑤ $\langle \vec{q}|\tilde{\phi}(0)|0\rangle = 1$ $\qquad |\vec{q}\rangle =$ 1-meson state
⑥ $\langle \vec{p}|\tilde{\psi}(0)|0\rangle = 1$ $\qquad |\vec{p}\rangle =$ 1-antinucleon state.

correct normalization for LSZ reduction formula

A few comments are in order:

- Note that we have chosen to separate out the canonical kinetic terms $\frac{1}{2}(\partial_\mu\tilde{\Phi})^2$ and $|\partial_\mu\tilde{\Psi}|^2$ from the counterterms. In general, this will require rescaling the fields. The fields which really have canonical kinetic term are ϕ and ψ, not $\tilde{\phi}$ and $\tilde{\psi}$. They are related by:

$$\frac{1}{2}(1+B)(\partial_\mu\tilde{\phi})^2 - \frac{1}{2}(\mu^2+C)\tilde{\phi}^2 + A\tilde{\phi} + \text{const.}$$
$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu_0^2}{2}\phi^2$$

and similarly for ψ.

The field ϕ satisfies $[\phi(t,\vec{x}), \dot{\phi}(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$.

The field $\tilde{\phi}$ satisfies $[\tilde{\phi}(t,\vec{x}), \dot{\tilde{\phi}}(t,\vec{y})] = (1+B)^{-1} i\delta^3(\vec{x}-\vec{y})$

- We added a term linear in $\tilde{\phi}$ even though there is no linear term in the original form of the Lagrangian. In principle, we don't know that any real monomial in the fields does not require a counterterm. We will see that sometimes this procedure works with only a finite number of counterterms. Such theories are predictive, and are called <u>renormalizable</u>. Theories for which this procedure fails are called <u>nonrenormalizable</u>. Nonrenormalizable theories will still prove valuable to think about in the context of <u>effective field theory</u>.

- The <u>renormalization conditions</u> ①, ②, ③ express terms in the Lagrangian in terms of the <u>physical</u> masses and coupling, but we should define what we mean by those physical quantities in terms of a calculation that can be compared w/ experiment.

We will clarify these points soon enough. First, we turn to:

<u>Renormalization of Spinor Theories</u>

Consider a real pseudoscalar field coupled to a Dirac spinor field with bare Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu_0^2\phi^2 + \bar{\psi}(i\not{\partial} - m_0)\psi - \lambda_0\phi^4$$
$$- g_0\bar{\psi}i\gamma_5\psi\phi$$

Note that we have added a term proportional to ϕ^4. Unlike the scalar theory considered previously with Dirac spinors ψ we will see that even if the <u>physical</u> ϕ^4 coupling vanishes a counterterm for it will be required.

We expect to be able to decompose the Lagrangian into a physical part and counterterms:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{\mu^2}{2}\tilde{\phi}^2 + \tilde{\bar{\Psi}}(i\partial\!\!\!/ - m)\tilde{\Psi} - \lambda\tilde{\phi}^4$$
$$- g\tilde{\bar{\Psi}} i\gamma_5 \tilde{\Psi}\tilde{\phi}$$
violates parity
$$\cancel{+ A\tilde{\phi}} + \frac{B}{2}(\partial_\mu\tilde{\phi})^2 - \frac{C}{2}\tilde{\phi}^2 + D\tilde{\bar{\Psi}} i\partial\!\!\!/ \tilde{\Psi} - E\tilde{\bar{\Psi}}\tilde{\Psi}$$
$$- F\tilde{\bar{\Psi}} i\gamma_5 \tilde{\Psi}\tilde{\phi} - G\tilde{\phi}^4$$

Φ is a pseudoscalar so we don't need to add a counterterm linear in $\tilde{\phi}$ since it would violate parity.

Also, since ϕ is pseudoscalar the condition $\boxed{\langle 0|\phi|0\rangle = 0}$ is satisfied automatically. We have one fewer condition to impose, and one fewer counterterms.

(We also have a new $\tilde{\phi}^4$ counterterm, but also one more physical coupling λ to match to experiment.)

For the pseudoscalar we can still impose the condition

$$\boxed{\langle \vec{q}|\tilde{\phi}(0)|0\rangle = 1.}$$

If the canonically normalized field ϕ does not satisfy this condition we just rescale it:

$$\phi = (1+B)^{1/2}\tilde{\phi}$$

A priori, we might have to rescale the different components of the Dirac spinor independently to impose an analogous condition. Let's see how this works.

For the free spinor, $\boxed{\langle 0|\underbrace{\psi(0)}_{\text{free field}}|\underbrace{r,p\rangle}_{\text{1-fermion state w/ momentum } p\text{, spin } r.} = u_p^r}$

It will be convenient to choose this as our renormalization condition, i.e. $\boxed{\langle 0|\tilde{\psi}(0)|r,p\rangle = u_p^r.}$

Is this choice consistent? Lorentz invariance and parity invariance hold the answer.

Consider the 1-fermion rest frame, and label

$J_z = +\frac{1}{2} \rightarrow r=1$

$J_z = -\frac{1}{2} \rightarrow r=2$

angular momentum operator

$$\langle 0|\psi(0)|1,p\rangle = \underbrace{\langle 0|e^{-iJ_z\theta}}_{\langle 0|}\ \underbrace{e^{iJ_z\theta}\psi(0)e^{-iJ_z\theta}}_{e^{-iL_z\theta}\psi(0)}\ \underbrace{e^{iJ_z\theta}|1,p\rangle}_{e^{i\theta/2}|1,p\rangle}$$

$$= e^{-iL_z\theta}e^{i\theta/2}\langle 0|\psi(0)|1,p\rangle$$

angular momentum generator

In the Weyl basis, $L_z = \frac{1}{2}\begin{pmatrix}\sigma_3 & \\ & \sigma_3\end{pmatrix}$, which restricts $\langle 0|\psi(0)|1,p\rangle$ to be of the form:

$$\boxed{\langle 0|\psi(0)|1,p\rangle = \begin{pmatrix} a \\ 0 \\ b \\ 0\end{pmatrix}}$$

If the theory also has a parity invariance (like our theory does) then $P\,\psi(0)\,P^{-1} = \gamma^0\psi(0)$

$\underset{\substack{\uparrow \\ \text{Parity}}}{P}\,|1,\underset{\substack{\nwarrow \\ \text{momentum}}}{p}\rangle = |1,p\rangle$ in <u>rest frame.</u>

Then $\langle 0|\psi(0)|1,p\rangle = \underbrace{\langle 0|P^{-1}}_{\langle 0|}\underbrace{P\psi(0)P^{-1}}_{\gamma^0\psi(0)}\underbrace{P|1,p\rangle}_{|1,p\rangle}$

$$= \gamma^0\langle 0|\psi(0)|1,p\rangle$$

In the Weyl basis $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so

$\langle 0|\psi(0)|1,p\rangle$ is further restricted to be of the form:

$$\boxed{\langle 0|\psi(0)|1,p\rangle = \begin{pmatrix} a \\ 0 \\ a \\ 0 \end{pmatrix}}$$

This means that whatever the effect of interactions on $\langle 0|\psi(0)|1,p\rangle$, we can always rescale $\psi(x)$ such that $\langle 0|\hat{\psi}(0)|1,p\rangle = u_p^1$, ie by choosing $\hat{\psi}(x) = \psi(x)\frac{\sqrt{m}}{a}$.

The same argument works for the $J_z = -\frac{1}{2}$ state $|2,p\rangle$. Hence, by rescaling the components of ψ we can choose $\langle 0|\hat{\psi}(0)|r,p\rangle = u_p^{(r)}$, as desired, and the reorganized Lagrangian in terms of the renormalized fields $\hat{\phi}$ and $\hat{\psi}$ is as written earlier.

Spinor Renormalization in Parity Non-Conserving Theories

Two things change in the absence of parity:

1) A more general set of counterterms is required (eg we used parity to argue there was no counterterm $A\tilde{\phi}$.)
   ↖ if $\tilde{\phi}$ is pseudoscalar

2) The renormalized field which satisfies $\langle 0|\tilde{\psi}(0)|r,p\rangle = u_p^r$ is no longer a simple rescaling of the field components.

If there is no parity invariance then we only know

$$\langle 0|\psi(0)|1,p\rangle = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}.$$

γ_5 commutes w/ Lorentz transformations, so $\gamma_5\psi$ transforms like ψ under Lorentz transformations.

$$\langle 0|\gamma_5\psi(0)|1,\underset{\text{at rest}}{p}\rangle = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -b \\ 0 \end{pmatrix}$$

So, whatever the interactions do to $\langle 0|\psi(0)|1,p\rangle$, we can always choose a renormalized field

$$\boxed{\tilde{\psi}(x) = \frac{\psi(x)(b+a) + \gamma_5\psi(x)(b-a)}{2ab}\sqrt{m}}$$ such that

$$\langle 0|\tilde{\psi}(0)|r,p\rangle = u_p^r.$$

Then renormalized pert. th. works as usual.