The Dirac Form Factor and Infrared Divergences

The renormalized electron vertex function has the form,

$$-i\tilde{\Gamma}^{\mu}(p,p') = -ie\gamma^{\mu}F_1(q^2) + \frac{\sigma^{\mu\nu}}{2m}q_\nu\, e F_2(q^2).$$

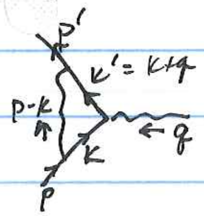
As you will show for homework, at one-loop, the Dirac form factor $F_1(q^2)$ may be written,

$$\boxed{\begin{aligned} F_1(q^2) = 1 + \frac{e^2}{8\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \delta(x+y+z-1) \\ \Bigg[\log\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy} + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy} \\ - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2}\Bigg] + \mathcal{O}(e^4). \end{aligned}}$$

The log term includes a counterterm that cancelled a UV divergence and helped set $F_1(0) = 1$.

The Feynman parameter integrals of the remaining terms diverge from the region $x \approx y \approx 0$, $z \approx 1$.
Let's first trace where this problem came from.

The denominator in the Feynman integral before doing the momentum integration was:



$$D = \left[x(k^2 - m^2) + y(k'^2 - m^2) + z(k-p)^2 + (x+y+z)i\epsilon\right]^3$$

When $x \approx y \approx 0$, $z \approx 1$, the denominator D vanishes when the loop momentum k satisfies $k - p \approx 0$. This is when the internal photon line has vanishing momentum, so it is refered to as an <u>infrared divergence</u>.

To regulate the infrared divergence we can introduce a <u>photon mass</u> parameter μ, which we will eventually take to zero. The factor $(k-p)^2 z$ in D comes from the photon propagator so we replace it by $[(k-p)^2 - \mu^2] z$. The net result is to add the term $\mu^2 z$ to the denominators in $F_1(q^2)$.

Considering the terms which diverge when $x \approx y \approx 0$, $z \approx 1$,

$$F_1(q^2) \sim 1 + \frac{e^2}{8\pi^2} \int_0^1 dz \int_0^{1-z} dy \left[\frac{-2m^2 + q^2}{m^2(1-z)^2 - q^2 y(1-z-y) + \mu^2} - \underbrace{\frac{-2m^2}{m^2(1-z)^2 + \mu^2}}_{\delta F_1(0)} \right]$$

Change variables $\xi = \frac{y}{1-z}$, $w = 1 - z$; $dz\,dy = \frac{1}{2} d\xi \, d(w^2)$

$$F_1(q^2) = 1 + \frac{e^2}{8\pi^2} \int_0^1 d\xi \cdot \frac{1}{2} \int_0^1 d(w^2) \left[\frac{-2m^2 + q^2}{w^2(m^2 - q^2\xi(1-\xi)) + \mu^2} - \frac{-2m^2}{m^2 w^2 + \mu^2} \right]$$

$$F_1(q^2) \simeq 1 + \frac{e^2}{8\pi^2}\int_0^1 d\xi \cdot \frac{1}{2}\left[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)} \log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right) + 2\log\frac{m^2}{\mu^2}\right]$$

If $q^2 < 4m^2$ the divergence as $\mu \to 0$ is insensitive to the numerators in the logs.

Details unimportant as $\mu \to 0$.

$$\boxed{F_1(q^2) \sim 1 - \frac{e^2}{8\pi^2}\left[\int_0^1 \frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)}\, d\xi - 1\right] \log\left(\frac{f(q^2 \text{ or } m^2)}{\mu^2}\right)}$$

We have succeeded in isolating the IR divergence. The effect of $F_1(q^2)$ on the cross section for scattering off a potential is just the replacement $e \to e\, F_1(q^2)$.

$$\boxed{\frac{d\sigma}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{tree}\\ \text{level}}}\left[1 - \frac{e^2}{4\pi^2}\left(\int_0^1 \frac{m^2 - q^2/2}{m^2-q^2\xi(1-\xi)}\, d\xi - 1\right) \log\left(\frac{f(q^2 \text{ or } m^2)}{\mu^2}\right)\right]}$$

The physical reason for the IR divergence is that any experiment can only measure emitted photons down to some energy E_{min}. When using a cross section to count events, you should add the cross sections for (that event + photon w/ $E < E_{min}$), + (that event + 2 photons w/ $E < E_{min}$), etc. This was suggested by Bloch and Nordsieck in 1937 (before relativistic perturbation theory existed).

Note that you have to add the cross sections, not the amplitudes, for all of these events.

We will study how the emission of low energy photons cancels the IR divergence in $F_1(q^2)$ in the limit $-q^2 \gg m^2$.

Then
$$\boxed{F_1(q^2)} \simeq 1 - \frac{e^2}{8\pi^2}\int_0^1 d\xi \, \frac{-q^2/2}{-q^2\xi(1-\xi)+m^2} \log\left(\frac{-q^2}{\mu^2}\right)$$
$$\boxed{\simeq 1 - \frac{e^2}{8\pi^2} \underbrace{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right)}_{\text{"Sudakov double log"}}}$$

We will find the same IR-divergent double log will appear in the cross section for emission of soft photons, and will cancel the IR divergence from $F_1(q^2)$.