Phys 721 F'23 Problem Set 7 Solutions

1. a) $\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu + b\, \partial_\mu A_\nu \partial^\nu A^\mu\right) + c A_\mu A^\mu$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\boxed{-\partial_\mu\left(\partial^\mu A^\nu + b\,\partial^\nu A^\mu\right) - 2c A^\nu = 0}$$

b) $A^\mu(x) = \varepsilon^\mu(k) e^{-ik\cdot x}$

$$\boxed{-(-ik_\mu)\left(-ik^\mu \varepsilon^\nu + b(-ik^\nu)\varepsilon^\mu\right) e^{-ik\cdot x} - 2c\,\varepsilon^\nu e^{-ik\cdot x} = 0}$$

c) $\varepsilon^\mu \propto k^\mu \rightarrow +k^2 k^\nu + b k^2 k^\nu - 2c k^\nu = 0$

$$-k^\nu\left(-k^2(1+b) + 2c\right) = 0$$

$$k^2 = \boxed{\frac{+2c}{1+b} \equiv m_L^2}$$

d) $\varepsilon^\mu k_\mu = 0 \rightarrow +k^2 \varepsilon^\nu - 2c\,\varepsilon^\nu = 0$

$$k^2 = \boxed{+2c \equiv m_T^2}$$

e) $m_L^2 \rightarrow \infty$ if $b \rightarrow -1$

$$\rightarrow \mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) + \frac{m_T^2}{2} A_\mu A^\mu$$

$$= \boxed{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m_T^2}{2} A_\mu A^\mu}$$

f) Taking a derivative of the Euler-Lagrange eqns gives:

$$\partial_\nu \partial_\mu \underbrace{\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)}_{F^{\mu\nu}} + m_T^2 \partial_\nu A^\nu = 0$$

Using the antisymmetry of $F^{\mu\nu}$ and the equality of mixed partial derivatives, the first term vanishes $\rightarrow \boxed{m_T^2 \partial_\nu A^\nu = 0}$