Phys 721 F'23 Problem Set 6 Solutions

1. $\boxed{\gamma^5\gamma^\mu\gamma^5} = -\gamma^5\gamma^5\gamma^\mu \boxed{= -\gamma^\mu}$

By cyclicity of the trace,

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu = \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^5$$
$$= -\mathrm{Tr}\,\gamma^\mu$$

Hence, $\boxed{\mathrm{Tr}\,\gamma^\mu = 0}$

$$\boxed{\mathrm{Tr}\,\not{a}} = a^\mu\,\mathrm{Tr}\,\gamma_\mu \boxed{= 0}$$

$$\boxed{\mathrm{Tr}\,\not{a}\not{b}} = a^\mu b^\nu\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu$$
$$= a^\mu b^\nu \tfrac{1}{2}\mathrm{Tr}\{\gamma_\mu,\gamma_\nu\}$$
$$= a^\mu b^\nu \cdot \eta_{\mu\nu}\,\mathrm{Tr}\,\mathbb{1}_{4\times4}$$
$$= 4a^\mu b^\nu \eta_{\mu\nu}$$
$$\boxed{= 4a\cdot b}$$

$$\mathrm{Tr}\,\not{a}\not{b}\not{c} = a^\mu b^\nu c^\lambda\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\lambda$$
$$= a^\mu b^\nu c^\lambda\,\mathrm{Tr}\,(\gamma^5)^2\gamma_\mu\gamma_\nu\gamma_\lambda$$
$$= a^\mu b^\nu c^\lambda\,\mathrm{Tr}\,\gamma^5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma^5 \quad \text{(cyclicity)}$$
$$= -a^\mu b^\nu c^\lambda\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\lambda \quad \text{(anticommuting } \gamma^5 \text{ past other } \gamma\text{'s)}$$
$$= -\mathrm{Tr}\,\not{a}\not{b}\not{c}$$

Hence, $\boxed{\mathrm{Tr}\,\not{a}\not{b}\not{c} = 0}$

$$\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d} = \mathrm{Tr}\,\not{b}\not{c}\not{d}\not{a} \quad \text{(cyclicity)}$$
$$\rightarrow \mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d} = \tfrac{1}{2}\left[\mathrm{Tr}\,\{\not{a},\not{b}\}\not{c}\not{d} - \mathrm{Tr}\,\not{b}\{\not{a},\not{c}\}\not{d} + \mathrm{Tr}\,\not{b}\not{c}\{\not{a},\not{d}\}\right]$$

$$\{\not{a},\not{b}\} = a_\mu b_\nu\{\gamma^\mu,\gamma^\nu\} = 2a\cdot b\,\mathbb{1}$$
$$\rightarrow \boxed{\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d}} = \tfrac{1}{2}\left[2a\cdot b\,\mathrm{Tr}\,\not{c}\not{d} - 2a\cdot c\,\mathrm{Tr}\,\not{b}\not{d} + 2a\cdot d\,\mathrm{Tr}\,\not{b}\not{c}\right]$$
$$\boxed{= 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]}$$

$\mathrm{Tr}\,\gamma^5 = i\,\mathrm{Tr}\,\gamma^0\gamma^1\gamma^2\gamma^3 = i\,\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d}$ with

$a_\mu = (1,0,0,0)$, $b_\mu = (0,1,0,0)$, $c_\mu = (0,0,1,0)$, $d_\mu = (0,0,0,1)$

$$\boxed{\mathrm{Tr}\,\gamma^5} = 4i\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right] \boxed{=0}$$

$$\mathrm{Tr}\,\gamma^5\not{a} = a_\mu\,\mathrm{Tr}\,\gamma^5\gamma^\mu = a_\mu\,\mathrm{Tr}\,\gamma^\mu\gamma^5 \quad \text{(cyclicity)}$$
$$= -a_\mu\,\mathrm{Tr}\,\gamma^5\gamma^\mu \quad \text{(anticommuting } \gamma^5\text{)}$$

$$\rightarrow \boxed{\mathrm{Tr}\,\gamma^5\not{a} = 0}$$

$\mathrm{Tr}\,\gamma^5\not{a}\not{b} = \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\,a_\mu b_\nu$

Pick a γ-matrix $\gamma^\alpha \neq \gamma^\mu, \gamma^\nu, \gamma^5$. $(\gamma^\alpha)^2 = \pm\mathbb{1}$.

$$\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu = \pm\,\mathrm{Tr}\,(\gamma^\alpha)^2\gamma^5\gamma^\mu\gamma^\nu = \pm\,\mathrm{Tr}\,\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha \quad \text{(cyclicity)}$$
$$= \mp\,\mathrm{Tr}\,(\gamma^\alpha)^2\gamma^5\gamma^\mu\gamma^\nu \quad \text{(anticommuting } \gamma^\alpha \text{ past other } \gamma\text{'s)}$$
$$= -\,\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu$$

$$\rightarrow \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu = 0, \text{ and } \boxed{\mathrm{Tr}\,\gamma^5\not{a}\not{b} = 0}$$

$$\mathrm{Tr}\,\gamma^5\not{a}\not{b}\not{c} = a_\mu b_\nu c_\rho\,\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho$$
$$= a_\mu b_\nu c_\rho\,\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5 \quad \text{(cyclicity)}$$
$$= -a_\mu b_\nu c_\rho\,\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho \quad \text{(anticommuting } \gamma^5\text{)}$$
$$= -\,\mathrm{Tr}\,\gamma^5\not{a}\not{b}\not{c}$$

$$\rightarrow \boxed{\mathrm{Tr}\,\gamma^5\not{a}\not{b}\not{c} = 0}$$

$\mathrm{Tr}\,\gamma^5 \not{a}\not{b}\not{c}\not{d} = a_\mu b_\nu c_\rho d_\sigma \,\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

If μ,ν,ρ,σ are not all different, then e.g.

if $\rho=\sigma$, $\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \pm\,\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu = 0$.

If $\mu\neq\nu$, etc. then by anticommutativity $\{\gamma^\mu,\gamma^\nu\}=0$, etc.

$\rightarrow$ $\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ completely antisymmetric in μ,ν,ρ,σ.

Hence, $\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = c\,\epsilon^{\mu\nu\rho\sigma}$ for some c.

$$\begin{aligned}\mathrm{Tr}\,\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3 &= i\,\mathrm{Tr}\,(\gamma^0\gamma^1\gamma^2\gamma^3)^2 \\ &= i\,\mathrm{Tr}\,(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2 \quad \text{(anticommute } \gamma\text{'s)} \\ &= i\,\mathrm{Tr}\,\mathbb{1}(-1)(-1)(-1) \\ &= -4i\end{aligned}$$

$\rightarrow$ $\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4i\,\epsilon^{\mu\nu\rho\sigma}$ with $\epsilon^{0123}=+1$.

$\rightarrow$ $\boxed{\mathrm{Tr}\,\gamma^5 \not{a}\not{b}\not{c}\not{d} = -4i\,\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma}$