PHYS 721 F'23 Problem Set 5 Solutions

1. a) $P_L^2 = \frac{1}{4}(1-\gamma^5)(1-\gamma^5) = \frac{1}{4}\left(1+(\gamma^5)^2 - 2\gamma^5\right)$

$= \frac{1}{4}\left(1+1-2\gamma^5\right) = \frac{1}{2}(1-\gamma^5) = P_L$

$P_R^2 = \frac{1}{4}(1+\gamma^5)(1+\gamma^5) = \frac{1}{4}(1+(\gamma^5)^2+2\gamma^5)$

$= \frac{1}{4}(1+1+2\gamma^5) = \frac{1}{2}(1+\gamma^5) = P_R$

$P_L P_R = \frac{1}{4}(1-\gamma^5)(1+\gamma^5) = \frac{1}{4}\left(1-(\gamma^5)^2\right) = \frac{1}{4}(1-1) = 0$

$P_R P_L = \frac{1}{4}(1+\gamma^5)(1-\gamma^5) = \frac{1}{4}\left(1-(\gamma^5)^2\right) = \frac{1}{4}(1-1) = 0$

b) Under a Lorentz transformation,

$\psi_L(x) = P_L \psi(x) \to P_L S(\Lambda)\psi(\Lambda^{-1}x)$

Since $[\gamma^5, S(\Lambda)] = 0$, $[P_L, S(\Lambda)] = 0$.

Hence, $\psi_L(x) \to S(\Lambda) P_L \psi(\Lambda^{-1}x) = S(\Lambda)\psi_L(\Lambda^{-1}x)$

Similarly, $\psi_R(x) \to S(\Lambda)\psi_R(\Lambda^{-1}x)$

c) $0 = P_L(i\partial\!\!\!/ - m)\psi = \frac{1}{2}(1-\gamma^5)(i\partial\!\!\!/ - m)\psi$

$= \left(i\partial\!\!\!/ \frac{(1+\gamma^5)}{2} - m\frac{(1-\gamma^5)}{2}\right)\psi$

$= i\partial\!\!\!/ P_R\psi - m P_L\psi$

$= i\partial\!\!\!/ \psi_R - m\psi_L$

$0 = P_R(i\partial\!\!\!/ - m)\psi = \frac{1}{2}(1+\gamma^5)(i\partial\!\!\!/ - m)\psi$

$= \left(i\partial\!\!\!/ \frac{(1-\gamma^5)}{2} - m\frac{(1+\gamma^5)}{2}\right)\psi$

$= i\partial\!\!\!/ \psi_L - m\psi_R$

d) $m \to 0$: $\left.\begin{array}{l} i\partial\!\!\!/ \psi_R = 0 \\ i\partial\!\!\!/ \psi_L = 0 \end{array}\right\}$ ψ_R and ψ_L are decoupled if $m=0$.

2. a) $i\bar{\sigma}^\mu \partial_\mu \chi - im\sigma^2 \chi^* = 0$, $\quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$

Recall that in the Weyl basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad S \approx \begin{pmatrix} 1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2} & 0 \\ 0 & 1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2} \end{pmatrix}$$

$$S^{-1} \approx \begin{pmatrix} 1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2} & 0 \\ 0 & 1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2} \end{pmatrix}$$

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

$$\Rightarrow \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = \Lambda^\mu{}_\nu \bar{\sigma}^\nu$$

(to order θ^2, β^2)

Multiplying on the right by $\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$,

$$(*) \quad \bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = \Lambda^\mu{}_\nu \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\nu$$

Then, Lorentz-transforming the Majorana eqn,

$$i\bar{\sigma}^\mu \partial_\mu \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\chi(\underbrace{\Lambda^{-1}x}_{\equiv x'}) - im\sigma^2\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\chi^*(\underbrace{\Lambda^{-1}x}_{\equiv x'})$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha{}_\mu \partial_\alpha \chi(x') - im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\chi^*(x')$$

(using $\sigma^2\vec{\sigma}^* = -\vec{\sigma}\sigma^2$)

$$= i\underbrace{(\Lambda^{-1})^\alpha{}_\mu \Lambda^\mu{}_\beta}_{\delta^\alpha{}_\beta}\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\beta \partial_\alpha \chi(x') - im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\chi^*(x')$$

(using $(*)$)

$$= \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\left[i\bar{\sigma}^\beta \partial_\beta \chi(x') - im\sigma^2\chi^*(x')\right] = 0$$

Hence, the Majorana eqn is covariant under Lorentz transformations if χ transforms as a left-handed spinor.

$i\bar{\sigma}^\mu \partial_\mu \chi - im\sigma^2 \chi^* = 0$. Complex conjugating,

$-i\bar{\sigma}^{\mu *}\partial_\mu \chi^* + im\sigma^{2*}\chi = 0$

Use $\sigma^{2*} = -\sigma^2$, $(\sigma^2)^2 = 1$, $\sigma^\nu \sigma^2 = \sigma^2 \bar{\sigma}^{\nu *}$,
and $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2\eta^{\mu\nu}$

$$\Rightarrow i\sigma^\nu \bar{\sigma}^\mu \partial_\nu \partial_\mu \chi - im\sigma^\nu \sigma^2 \partial_\nu \chi^* = 0$$
$$= \frac{i}{2}\{\sigma^\nu, \bar{\sigma}^\mu\}\partial_\nu\partial_\mu \chi - im\sigma^2 \bar{\sigma}^{\nu *}\partial_\nu \chi^*$$
$$= i\partial_\mu \partial^\mu \chi - im\sigma^2(-m\sigma^2\chi)$$
$$= i\partial_\mu\partial^\mu \chi + im^2 \chi$$

$$\rightarrow \boxed{\partial_\mu \partial^\mu \chi + m^2 \chi = 0}$$

b) $S = \int d^4x \left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$

$$S^* = \int d^4x \left[-i\left(-\chi^T\bar{\sigma}^*\cdot\partial\chi^*\right) - \frac{im}{2}\left(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right]$$

$$= \int d^4x \left[-i\,\partial_\mu\chi^T\bar{\sigma}^{\mu *}\chi^* + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

(integrating by parts)

$$= \int d^4x \left[i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

(transposing the first term and using $\bar{\sigma}^{\mu\dagger} = \bar{\sigma}^\mu$)

$= S$.

$$0 = \delta S = \int d^4x \Big[(\delta\chi^*)^T i\bar{\sigma}\cdot\partial\chi + i\chi^\dagger\bar{\sigma}\cdot\partial\delta\chi$$

transpose and integrate by parts.

$$+ \frac{im}{2}\Big((\delta\chi^T)\sigma^2\chi + \chi^T\sigma^2\delta\chi - (\delta\chi^*)^T\sigma^2\chi^* - \chi^\dagger\sigma^2\delta\chi^* \Big)\Big]$$

transpose this term   transpose this term

$$= \int d^4x \Big[(\delta\chi^*)^T \Big(i\bar{\sigma}\cdot\partial\chi - \frac{im}{2}\sigma^2\chi^* + \frac{im}{2}(\sigma^2)^T\chi^* \Big)$$

$$+ (\delta\chi)^T \Big(i(\bar{\sigma})^T\cdot\partial\chi^* + \frac{im}{2}\sigma^2\chi - \frac{im}{2}(\sigma^2)^T\chi \Big)\Big]$$

Use $(\sigma^2)^T = -\sigma^2$, $\bar{\sigma}^T = \bar{\sigma}^*$, and set the variations to zero

$$\Rightarrow \quad i\bar{\sigma}\cdot\partial\chi - im\,\sigma^2\chi^* = 0 \qquad (\delta\chi^* \text{ variation})$$

$$i\bar{\sigma}^*\cdot\partial\chi^* + im\,\sigma^2\chi = 0 \qquad (\delta\chi \text{ variation})$$

Using $(\sigma^2)^* = -\sigma^2$, these two eqs. of motion are complex conjugates of one another.

c) $-\left((i\partial\!\!\!/ + im'\gamma_5) + m\right)\left((i\partial\!\!\!/ + im'\gamma_5) - m\right)\psi = 0$

$$-\left((i\partial\!\!\!/ + im'\gamma_5)^2 - m^2\right)\psi = 0$$

$$= -\left(-\partial\!\!\!/\partial\!\!\!/ - (m')^2(\gamma_5)^2 - \cancel{\{\gamma_5, \partial\!\!\!/\}}^{0} - m^2\right)\psi$$

$$= \left(\partial_\mu \partial^\mu + (m')^2 + m^2\right)\psi$$

$$\Rightarrow \boxed{\left(\partial_\mu \partial^\mu + \underbrace{(m')^2 + m^2}_{= M^2}\right)\psi = 0}$$