Phys 721 F'23 Problem Set 4 Solutions

1. a) $\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x} \right)$

$$H = \int d^3x \left[(\partial_0\phi^\dagger) \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)} + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} \partial_0\phi - \mathcal{L} \right]$$

$$= \int d^3x \left[(\partial_0\phi^\dagger)(\partial_0\phi) + \nabla\phi^\dagger\cdot\nabla\phi + m^2\phi^\dagger\phi \right]$$

$$= \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \Big[(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2) e^{i(k-k')\cdot x} a_k^\dagger a_{k'}$$

$$+ (\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2) e^{-i(k-k')\cdot x} b_k b_{k'}^\dagger$$

$$+ (-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2) e^{i(k+k')\cdot x} a_k^\dagger b_{k'}$$

$$+ (-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2) e^{-i(k+k')\cdot x} b_k^\dagger a_{k'} \Big]$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \Big[(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2)(2\pi)^3\delta^3(\vec{k}-\vec{k}') e^{i(\omega_k-\omega_{k'})t} a_k^\dagger a_{k'}$$

$$+ (\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2)(2\pi)^3\delta^3(\vec{k}-\vec{k}') e^{-i(\omega_k-\omega_{k'})t} b_k b_{k'}^\dagger$$

$$+ (-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2)(2\pi)^3\delta^3(\vec{k}+\vec{k}') e^{i(\omega_k+\omega_{k'})t} a_k^\dagger b_{k'}$$

$$+ (-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2)(2\pi)^3\delta^3(\vec{k}+\vec{k}') e^{-i(\omega_k+\omega_{k'})t} b_k^\dagger a_{k'} \Big]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \cdot 2\omega_k^2 \left[a_k^\dagger a_k + b_k b_k^\dagger \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \omega_k \left[a_k^\dagger a_k + b_k b_k^\dagger \right]$$

b) $$\vec{P} = -\int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \nabla\phi + (\nabla\phi^+) \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^+} \right]$$

$$= -\int d^3x \left[(\partial_0 \phi^+) \nabla\phi + (\nabla\phi^+) \partial_0 \phi \right]$$

$$= -\int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \Big[\omega_k \vec{k}' \Big(-a_k^+ a_{k'} e^{i(k-k')\cdot x} - b_k b_{k'}^+ e^{-i(k-k')\cdot x}$$

$$+ a_k^+ b_{k'}^+ e^{i(k+k')\cdot x} + b_k a_{k'} e^{-i(k+k')\cdot x} \Big) \Big]$$

+ hermitian conjugate

$$= -\int \frac{d^3k\, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \Big[-\omega_k \vec{k}' \Big(a_k^+ a_{k'} e^{i(\omega_k - \omega_{k'})t} + b_k b_{k'}^+ e^{-i(\omega_k - \omega_{k'})t} \Big) (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$+ \omega_k \vec{k}' \Big(a_k^+ b_{k'}^+ e^{i(\omega_k + \omega_{k'})t} + b_k a_{k'} e^{-i(\omega_k + \omega_{k'})t} \Big) (2\pi)^3 \delta^3(\vec{k}+\vec{k}') \Big]$$

+ hermitian conjugate

from hermitian conjugate

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \cdot \omega_k \Big[\vec{k} \left(a_k^+ a_k + b_k b_k^+ \right) \times 2$$

$$- \vec{k} \left(a_{\vec{k}}^+ b_{-\vec{k}}^+ e^{2i\omega_k t} + b_{\vec{k}} a_{-\vec{k}} e^{-2i\omega_k t} \right)$$

$$- \vec{k} \left(b_{-\vec{k}} a_{\vec{k}} e^{-2i\omega_k t} + a_{-\vec{k}}^+ b_{\vec{k}}^+ e^{+2i\omega_k t} \right) \Big]$$

Change variables $\vec{k} \to -\vec{k}$ in the last line

$$\Rightarrow \boxed{\vec{P} = \int \frac{d^3k}{(2\pi)^3} \vec{k} \left(a_k^+ a_k + b_k b_k^+ \right)}$$

c) $[:H:, a_k^\dagger] = \int \frac{d^3k'}{(2\pi)^3} \omega_{k'} [a_{k'}^\dagger a_{k'}, a_k^\dagger]$

$$= \int \frac{d^3k'}{(2\pi)^3} \omega_{k'} a_{k'}^\dagger (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$= \omega_k a_k^\dagger$$

Similarly, $[:H:, b_k^\dagger] = \int \frac{d^3k'}{(2\pi)^3} \omega_{k'} [:b_{k'} b_{k'}^\dagger:, b_k^\dagger]$

$$= \int \frac{d^3k'}{(2\pi)^3} \omega_{k'} b_{k'}^\dagger (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$= \omega_k b_k^\dagger$$

Hence, $:H:(a_k^\dagger|0\rangle) = [:H:, a_k^\dagger]|0\rangle = \omega_k (a_k^\dagger|0\rangle)$

and $:H:(b_k^\dagger|0\rangle) = [:H:, b_k^\dagger]|0\rangle = \omega_k (b_k^\dagger|0\rangle)$,

where we have used $:H:|0\rangle = 0$.

Therefore, the states $a_k^\dagger|0\rangle$ and $b_k^\dagger|0\rangle$ are both eigenstates of H with eigenvalue $\omega_k > 0$ (above the ground state energy).

2. a) $\langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle\,\theta(x^0-y^0)$
$\qquad + \langle 0|\phi^\dagger(y)\,\phi(x)|0\rangle\,\theta(y^0-x^0)$

$$\partial^\mu_x \langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle = \langle 0|T[\partial^\mu_x\phi(x)\,\phi^\dagger(y)]|0\rangle + \delta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle\delta^{\mu 0} - \delta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle\delta^{\mu 0}$$

$$= \langle 0|T[\partial^\mu_x\phi(x)\,\phi^\dagger(y)]|0\rangle + \delta^{\mu 0}\underbrace{\delta(x^0-y^0)\langle 0|[\phi(x),\phi^\dagger(y)]|0\rangle}_{=0 \text{ by the ETCR's.}}$$

$$(\partial^\mu\partial^x_\mu + m^2)\langle 0|T[\phi(x)\,\phi^\dagger(y)]|0\rangle$$
$$= \langle 0|T[\underbrace{(\partial^\mu\partial^x_\mu + m^2)\phi(x)}_{=0 \text{ by the eqs of motion}}\,\phi^\dagger(y)]|0\rangle$$
$$+ \delta(x^0-y^0)\langle 0|\partial_0\phi(x)\,\phi^\dagger(y)|0\rangle - \delta(x^0-y^0)\langle 0|\phi^\dagger(y)\,\partial_0\phi(x)|0\rangle$$

$$= \delta(x^0-y^0)\langle 0|[\partial_0\phi(x),\phi^\dagger(y)]|0\rangle$$
$$= \delta(x^0-y^0)\left(-i\,\delta^3(\vec{x}-\vec{y})\right)$$

$$\boxed{(\partial^\mu\partial^x_\mu + m^2)\langle 0|T[\phi(x)\,\phi^\dagger(y)]|0\rangle = -i\,\delta^4(x-y)}$$

b) $\langle 0|\phi(x)\,\phi(y)|0\rangle$

$$= \langle 0|\int \frac{d^3k\,d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right)\left(a_{k'}e^{-ik'\cdot y} + b_{k'}^\dagger e^{ik'\cdot y}\right)|0\rangle$$

$= 0$ using $\langle 0|b_k^\dagger = 0$, $a_{k'}|0\rangle = 0$, and $[a_k, b_{k'}^\dagger] = [b_k^\dagger, a_{k'}] = 0$.

Similarly, $\langle 0|\phi(y)\phi(x)|0\rangle = 0$. Hence,

$$\boxed{\langle 0|T(\phi(x)\,\phi(y))|0\rangle = 0}$$

Similarly, $\langle 0|\phi^+(x)\phi^+(y)|0\rangle$

$$= \langle 0| \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \left(a_k^+ e^{+ik\cdot x} + b_k e^{-ik\cdot x}\right)\left(a_{k'}^+ e^{ik'\cdot y} + b_{k'} e^{-ik'\cdot y}\right)|0\rangle$$

$= 0$. Hence,

$$\boxed{\langle 0|T(\phi^+(x)\phi^+(y))|0\rangle = 0}$$

$\langle 0|\phi(x)\phi^+(y)|0\rangle$

$$= \langle 0| \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \left(a_k e^{-ik\cdot x} + b_k^+ e^{ik\cdot x}\right)\left(a_{k'}^+ e^{ik'\cdot y} + b_{k'} e^{-ik'\cdot y}\right)|0\rangle$$

$$= \langle 0| \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} a_k a_{k'}^+ e^{-ik\cdot x + ik'\cdot y} |0\rangle$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \underbrace{\langle 0|[a_k, a_{k'}^+]|0\rangle}_{(2\pi)^3\delta^3(\vec{k}-\vec{k}')} e^{-ik\cdot x + ik'\cdot y}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)}$$

Similarly, $\langle 0|\phi^+(y)\phi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik\cdot(x-y)}$

→ $\langle 0|T(\phi(x)\phi^+(y))|0\rangle = \theta(x^0-y^0)\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} + \theta(y^0-x^0)\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik\cdot(x-y)}$

or

$$\boxed{\langle 0|T(\phi(x)\phi^+(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}}$$