Phys 721 F'23 Problem Set 3 Solutions

1. a) $e^{iHt} a_k e^{-iHt} = \sum_{n=0}^{\infty} \frac{(iHt)^n}{n!} a_k e^{-iHt}$

Using $[H, a_k] = -\omega_k a_k$,

$H a_k = a_k (H - \omega_k)$

Iterating, $H^2 a_k = H a_k (H-\omega_k) = a_k (H-\omega_k)^2$

$\vdots$

$H^n a_k = a_k (H-\omega_k)^n$

Hence, $\boxed{e^{iHt} a_k e^{-iHt}} = \sum_{n=0}^{\infty} a_k \frac{(itH - it\omega_k)^n}{n!} e^{-iHt}$

$= a_k e^{iHt - i\omega_k t} e^{-iHt}$

$\boxed{= a_k e^{-i\omega_k t}}$

Similarly, using $[H, a_k^+] = \omega_k a_k^+$,

$H^n a_k^+ = a_k^+ (H+\omega_k)^n$

$\boxed{e^{iHt} a_k^+ e^{-iHt}} = \sum_{n=0}^{\infty} a_k^+ \frac{(itH + it\omega_k)^n}{n!} e^{-iHt}$

$= a_k^+ e^{iHt + i\omega_k t} e^{-iHt}$

$\boxed{= a_k^+ e^{i\omega_k t}}$

Similarly, using $[\vec{P}, a_{\vec{k}}] = -\vec{k} a_{\vec{k}}$,

$\boxed{e^{-i\vec{P}\cdot\vec{x}} a_{\vec{k}} e^{i\vec{P}\cdot\vec{x}}} = \sum_{n=0}^{\infty} a_{\vec{k}} \frac{(-i\vec{P}\cdot\vec{x} + i\vec{k}\cdot\vec{x})^n}{n!} e^{i\vec{P}\cdot\vec{x}}$

$= a_{\vec{k}} e^{-i\vec{P}\cdot\vec{x} + i\vec{k}\cdot\vec{x}} e^{i\vec{P}\cdot\vec{x}} \boxed{= a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}}$

$$\boxed{e^{-i\vec{P}\cdot\vec{x}}\, a^\dagger_{\vec{k}}\, e^{i\vec{P}\cdot\vec{x}}} = \sum_{n\geq 0}^{\infty} a^\dagger_{\vec{k}} \frac{(-i\vec{P}\cdot\vec{x} - i\vec{k}\cdot\vec{x})^n}{n!} e^{i\vec{P}\cdot\vec{x}}$$

$$\boxed{= a^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}}$$

b) $e^{i(Ht-\vec{P}\cdot\vec{x})}\,\phi(0)\,e^{-i(Ht-\vec{P}\cdot\vec{x})}$

$$= e^{i(Ht-\vec{P}\cdot\vec{x})} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} (a_k + a_k^\dagger)\, e^{-i(Ht-\vec{P}\cdot\vec{x})}$$

Using the results of part (a) and $[H,\vec{P}]=0$,

$$\boxed{e^{i(Ht-\vec{P}\cdot\vec{x})}\,\phi(0)\,e^{-i(Ht-\vec{P}\cdot\vec{x})}}$$

$$= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right)$$

$$\boxed{= \phi(x)}$$

c) Using $P^\mu|0\rangle = 0$ and $[P^\mu, P^\nu]=0$,

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0| e^{-iP\cdot y} e^{iP\cdot x}\phi(0) e^{-iP\cdot x} e^{iP\cdot y}\phi(0)\underbrace{e^{-iP\cdot y}|0\rangle}_{|0\rangle}$$

$$= \langle 0|\underbrace{e^{iP\cdot(x-y)}\phi(0)e^{-iP\cdot(x-y)}}_{\phi(x-y)}\phi(0)|0\rangle$$

$$= \langle 0|\phi(x-y)\,\phi(0)|0\rangle$$

2. $\left.\frac{\partial \phi_1}{\partial \theta}\right|_{\theta=0} = \phi_2 , \quad \left.\frac{\partial \phi_2}{\partial \theta}\right|_{\theta=0} = -\phi_1 , \quad \mathcal{L} \to \mathcal{L}$

$$J^\mu = \sum_{a=1,2} \Pi_a^\mu \left.\frac{\partial \phi_a}{\partial \theta}\right|_{\theta=0} = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1$$

$$Q = \int d^3x \, J^0 = \int \left[(\partial_0 \phi_1)\phi_2 - (\partial_0 \phi_2)\phi_1\right] d^3x$$

$$= \int d^3x \frac{d^3k \, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \, i\omega_k \Big(\left(-a_k^1 e^{-ik\cdot x} + a_k^{1\dagger} e^{ik\cdot x}\right) \cdot \left(a_{k'}^2 e^{-ik'\cdot x} + a_{k'}^{2\dagger} e^{ik'\cdot x}\right)$$

$$- \left(-a_k^2 e^{-ik\cdot x} + a_k^{2\dagger} e^{ik\cdot x}\right)\left(a_{k'}^1 e^{-ik'\cdot x} + a_{k'}^{1\dagger} e^{ik'\cdot x}\right)\Big)$$

$$= \int \frac{d^3k \, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \, i\omega_k \Big(a_k^{1\dagger} a_{k'}^2 e^{it(\omega_k - \omega_{k'})} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$- a_k^1 a_{k'}^{2\dagger} e^{-it(\omega_k - \omega_{k'})} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$+ a_k^2 a_{k'}^{1\dagger} e^{-it(\omega_k - \omega_{k'})} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$- a_k^{2\dagger} a_{k'}^1 e^{it(\omega_k - \omega_{k'})} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

+ terms that cancel due to the δ-functions and commutation relations $[a_k^1, a_{k'}^2] = 0$, etc. $\Big)$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, i\omega_k \left(a_k^{1\dagger} a_k^2 + a_k^{2\dagger} a_k^1 - a_k^1 a_k^{2\dagger} - a_k^{2\dagger} a_k^1 \right)$$

Normal ordering, $\boxed{:Q: = \int \frac{d^3k}{(2\pi)^3} \, i\left(a_k^{1\dagger} a_k^2 - a_k^{2\dagger} a_k^1\right)}$

3.a) Consider an n-particle state $|n\rangle = C\, a^\dagger_{k_1} a^\dagger_{k_2} \cdots a^\dagger_{k_n} |0\rangle$, where C is a normalization constant.

$$N|n\rangle = \int \frac{d^3k}{(2\pi)^3} a^\dagger_k a_k\, C\, a^\dagger_{k_1} \cdots a^\dagger_{k_n} |0\rangle$$

Using $[a_k, a^\dagger_{k_i}] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}_i)$ n times to commute a_k all the way to the right,

$$\boxed{N|n\rangle} = \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 C\, a^\dagger_k \sum_{i=1}^{n} \left(\prod_{j \neq i} a^\dagger_{k_j} \right) \delta^3(\vec{k} - \vec{k}_i) |0\rangle$$

$$= \sum_{i=1}^{n} C\, a^\dagger_{k_i} \left(\prod_{j\neq i} a^\dagger_{k_j} \right) |0\rangle = C \sum_{i=1}^{n} \left(\prod_{j=1}^{n} a^\dagger_{k_j} \right) |0\rangle$$

$$= n\, C \prod_{j=1}^{n} a^\dagger_{k_j} |0\rangle \boxed{= n|n\rangle}$$

Hence, N counts the number of particles in the state.

b) Use $a_{\vec{k}} = \frac{1}{2} \int d^3x\, e^{ik\cdot x} \left(\sqrt{2\omega_k}\, \phi(x) + i \sqrt{\frac{2}{\omega_k}}\, \Pi(x) \right)$

$$a^\dagger_{\vec{k}} = \frac{1}{2} \int d^3x\, e^{-ik\cdot x} \left(\sqrt{2\omega_k}\, \phi(x) - i \sqrt{\frac{2}{\omega_k}}\, \Pi(x) \right)$$

$$\boxed{N = \int \frac{d^3k}{(2\pi)^3} \int d^3x\, d^3x'\, e^{-ik\cdot(x-x')} \left(\sqrt{2\omega_k}\, \phi(x) - i\sqrt{\frac{2}{\omega_k}}\, \Pi(x) \right) \left(\sqrt{2\omega_k}\, \phi(x') + i \sqrt{\frac{2}{\omega_k}}\, \Pi(x') \right)}$$

$k^0 = \omega_k$, $t = t'$

Using the equal time commutation relations the terms of the form $\phi\Pi$ and $\Pi\phi$ can be evaluated, yielding an unphysical divergence. This divergence simply cancels a similar divergence in the remaining terms.