Phys 721 F'23 Problem Set 2 Solutions

1. $$[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{16}\Big[[\gamma^\mu,\gamma^\nu],[\gamma^\rho,\gamma^\sigma]\Big]$$

Use $[\gamma^\mu,\gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = 2\gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}\mathbb{1}$, and $[2\eta^{\mu\nu}\mathbb{1}, \text{anything}] = 0$. Then,

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\frac{1}{4}[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma]$$

$$= -\frac{1}{4}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right)$$

$$= -\frac{1}{4}\left(-\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma + 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right)$$

$$= -\frac{1}{4}\left(\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right)$$

$$= -\frac{1}{4}\left(-\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu + 2\eta^{\sigma\nu}\gamma^\rho\gamma^\mu - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right)$$

$$= -\frac{1}{4}\Big(\cancel{\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu} - 2\eta^{\mu\sigma}\gamma^\rho\gamma^\nu + 2\eta^{\sigma\nu}\gamma^\rho\gamma^\mu - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma$$
$$+ 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \cancel{\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu}\Big)$$

Now use $\gamma^\rho\gamma^\nu = \frac{1}{2}[\gamma^\rho,\gamma^\nu] + \eta^{\rho\nu}$ to express each factor quadratic in γ's as a commutator:

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{4}\Big(\eta^{\mu\sigma}[\gamma^\rho,\gamma^\nu] - \eta^{\nu\sigma}[\gamma^\rho,\gamma^\mu] + \eta^{\rho\mu}[\gamma^\nu,\gamma^\sigma]$$
$$- \eta^{\rho\nu}[\gamma^\mu,\gamma^\sigma]\Big) + \text{terms that cancel.}$$

$$= i\left(\eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho} - \eta^{\rho\mu}S^{\nu\sigma} + \eta^{\rho\nu}S^{\mu\sigma}\right).$$

Hence, $S^{\mu\nu}$ satisfy the Lorentz algebra.

2. $\mathcal{L} = |\partial_\mu \psi|^2 - M^2|\psi|^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - g\psi^*\psi\phi - \lambda\phi^4$

a) $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \rightarrow \partial_\mu\partial^\mu \psi^* + M^2\psi^* + g\psi^*\phi = 0$

$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\right) - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \Rightarrow \partial_\mu\partial^\mu \psi + M^2\psi + g\psi\phi = 0$

$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \partial_\mu\partial^\mu \phi + m^2\phi + g\psi^*\psi + 4\lambda\phi^3 = 0$

b) $\psi \rightarrow e^{i\theta}\psi, \quad \psi^* \rightarrow e^{-i\theta}\psi^*$

$\frac{\partial \psi}{\partial \theta}\Big|_{\theta=0} = i\psi, \quad \frac{\partial \psi^*}{\partial \theta}\Big|_{\theta=0} = -i\psi^*$

$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(i\psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}(-i\psi^*)$

$= (\partial^\mu \psi^*)(i\psi) + (\partial^\mu \psi)(-i\psi^*)$

$= 2\,\mathrm{Im}\,(\psi^*\partial^\mu \psi)$

c) $H = \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}\partial_0\psi + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^*)}\partial_0\psi^* + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial_0\phi\right) - \mathcal{L}\right]$

$= \int d^3x \left[2|\partial_0\psi|^2 + (\partial_0\phi)^2 - \mathcal{L}\right]$

$= \int d^3x \left[|\partial_0\psi|^2 + |\nabla\psi|^2 + \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + M^2|\psi|^2 + \frac{m^2}{2}\phi^2 + g\psi^*\psi\phi + \lambda\phi^4\right]$

$P^i = -\int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}\partial_i\psi + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^*)}\partial_i\psi^* + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial_i\phi\right]$

$= -\int d^3x \left[\partial_0\psi^*\partial_i\psi + \partial_0\psi\,\partial_i\psi^* + \partial_0\phi\,\partial_i\phi\right]$

H is unbounded below ~~as~~ $M^2 + g\phi < 0$, $|\psi|^2 \rightarrow \infty$.