Physics 721, Fall 2023
Problem Set 2
Due Mpnday, September 25.

1. *The so(3,1) Algebra*

The generators of the rotation group in three dimensions, SO(3), satisfy the algebra $\left[T^a, T^b\right] = i \sum_c \epsilon^{abc}\, T^c$. The matrices $T^a = \sigma^a/2$, $a = 1, 2, 3$, form a representation of the algebra.

The analogous relations for the six generators of Lorentz transformations $J^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, with $J^{\mu\nu} = -J^{\nu\mu}$, are

$$[J^{\mu\nu}, J^{\rho\sigma}] = i \left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\right).$$

These commutation relations define the algebra so(3,1). Using the properties of the Dirac γ-matrices, show that the generators of Lorentz transformations in the Dirac spinor representation,

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right],$$

satisfy the commutation relations describing the Lorentz algebra.

2. *Scalar Fields with Interactions*

Consider a theory of a complex scalar field $\psi(x)$ and a real scalar field $\phi(x)$, with Lagrangian density,

$$\mathcal{L} = |\partial_\mu \psi|^2 - M^2\,|\psi|^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - g\,\psi^* \psi \phi - \lambda\,\phi^4$$

where M, m, g, and λ are constants.

a) What are the Euler-Lagrange equations for ψ, ψ^*, and ϕ?

b) What is the 4-vector current associated with the symmetry $\psi \to e^{i\theta}\psi$, $\psi^* \to e^{-i\theta}\psi^*$? What is the associated conserved charge?

c) What are the conserved energy and spatial momentum in terms of ψ and ϕ? Is the energy bounded below for some choice of signs of the constants in the Lagrangian density?