Physics 721, Fall 2023
Problem Set 1
Due Monday, September 18.

1. *The Dirac Equation*

a) In the absence of interactions, the Dirac equation for a particle at rest takes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = mc^2 \, \beta \, \Psi.$$

With the matrix β in the Dirac basis, find four independent solutions for $\Psi(t)$ with definite energy. You should find two solutions with positive energy and two with negative energy. (The energy is the eigenvalue of $i\hbar \, \partial/\partial t$.)

b) Including the coupling to electromagnetism, write the Dirac equation in terms of φ and χ, where

$$\Psi = e^{imc^2 t/\hbar} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \quad \text{in the Dirac basis.}$$

c) For the negative energy, nonrelativistic solutions to the Dirac equation, assume that φ and χ are slowly varying functions of time. For these solutions, write an approximate algebraic relation between φ and χ, and identify the large components of Ψ.

d) Assuming a weak, uniform, magnetic field $\mathbf{B}$, derive a differential equation involving only the large components of Ψ, and $\mathbf{B}$. Compare with the analogous equation for the positive-energy solutions described in class.

2. *Tensors*

Assume the matrix $\Lambda^\mu{}_\nu$ describes a Lorentz transformation, such that $x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$.

a) If $T^{\mu\nu}$ and $B^{\mu\nu}$ are tensors under Lorentz transformations, prove that $T^{\mu\nu} B_{\nu\mu}$ and $T^{\mu\nu} B_{\mu\nu}$ are Lorentz scalars.

b) How does $T^{\mu\nu} B_\mu{}^\alpha$ transform? What kind of tensor is this?

c) If $\phi(x)$ is a scalar field, show that $\partial_\mu \phi \, \partial^\mu \phi$ transforms as a scalar field.

d) Show that as a tensor under Lorentz transformations, the Minkowski tensor $\eta_{\mu\nu}$ is Lorentz invariant.

e) Show that if under a Lorentz transformation $x^\mu \to \sum_\nu \Lambda^\mu{}_\nu \, x^\nu$, then

$$x_\mu \to \sum_\nu (\Lambda^{-1})^\nu{}_\mu \, x_\nu .$$