Heisenberg Eqs. of Motion

The equal time commutation relations (ETCR's) are consistent with the Heisenberg equations of motion:

$$H = \int d^3x \ \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$$\partial_0 \phi(\vec{y},t) = i\left[H, \phi(\vec{y},t)\right]$$

$$= i\int d^3x \ \frac{1}{2}\left[\Pi(\vec{x},t)^2, \phi(\vec{y},t)\right]$$
$$= i\int d^3x \ \Pi(\vec{x},t)\left[\Pi(\vec{x},t), \phi(\vec{y},t)\right]$$
$$= i(-i)\int d^3x \ \Pi(\vec{x},t)\,\delta^3(\vec{x}-\vec{y}) = \Pi(\vec{y},t)$$

$$\rightarrow \boxed{\partial_0\phi(\vec{y},t) = \Pi(\vec{y},t)}$$

$$\partial_0 \Pi(\vec{y},t) = i\left[H, \Pi(\vec{y},t)\right]$$

$$= \frac{i}{2}\int d^3x \left[(\nabla\phi(\vec{x},t))^2 + m^2\phi(\vec{x},t)^2, \Pi(\vec{y},t)\right]$$
$$= i\int d^3x \Big(\nabla\phi(\vec{x},t)\cdot\left[\nabla\phi(\vec{x},t), \Pi(\vec{y},t)\right]$$
$$+ m^2\phi(\vec{x},t)\left[\phi(\vec{x},t), \Pi(\vec{y},t)\right]\Big)$$

$$= i\int d^3x \left(-\nabla^2\phi(\vec{x},t) + m^2\phi(\vec{x},t)\right)\left[\phi(\vec{x},t), \Pi(\vec{y},t)\right]$$
$$= \nabla^2\phi(\vec{y},t) - m^2\phi(\vec{y},t)$$

$$\Rightarrow \boxed{\partial_0 \Pi(\vec{y},t) = \nabla^2\phi(\vec{y},t) - m^2\phi(\vec{y},t)}$$

Combining the two eqs we recover the eq of motion

$$\partial_0^2\phi = \partial_0\partial_0\Pi = \nabla^2\phi - m^2\phi$$
$$\rightarrow \boxed{\partial_\mu\partial^\mu\phi + m^2\phi = 0}$$

Hilbert Space of the Scalar Field Hamiltonian

The free scalar field Hamiltonian describes a harmonic oscillator for each $\vec{k}$, so the Hilbert space is that of a harmonic oscillator for each $\vec{k}$.

The normal ordered Hamiltonian is

$$:H: = \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} \, a^\dagger_{\vec{k}} a_{\vec{k}}$$

which for simplicity we will just call H.

$$[H, a^\dagger_{k'}] = \int \frac{d^3k}{(2\pi)^3} \omega_k \, a^\dagger_k [a_k, a^\dagger_{k'}]$$

$$= \int \frac{d^3k}{(2\pi)^3} \omega_k \, a^\dagger_k (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$= \omega_{k'} a^\dagger_{k'}$$

$$[H, a_{k'}] = \int \frac{d^3k}{(2\pi)^3} \omega_k [a^\dagger_k, a_{k'}] a_k$$

$$= \int \frac{d^3k}{(2\pi)^3} \omega_k \left(-(2\pi)^3\right) \delta^3(\vec{k} - \vec{k}') \, a_k$$

$$= -\omega_{k'} a_{k'}$$

These commutators imply that $a^\dagger_k$ raises the energy of an energy eigenstate, and a_k lowers the energy.

The ground state satisfies $a_{\vec{k}}|0\rangle = 0 \ \forall \vec{k}$.
We dropped the ground state energy from H, ie. $H|0\rangle = 0$.

Using the commutation relations for $[H, a_k^\dagger]$ we can compute:

$$H\left(a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_n}^\dagger |0\rangle\right) = \left(\omega_{k_1} + \cdots + \omega_{k_n}\right) a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$$

The states $a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$ form the Hilbert space of the free scalar field. We will choose a normalization for the states shortly.

Spatial Momentum

The spatial momentum operator is the conserved charge due to spatial translation invariance.

$$\phi(\vec{x}, t) \to \phi(\vec{x} + \vec{a}, t) = \phi(\vec{x}, t) + \vec{a} \cdot \nabla\phi(\vec{x}, t) + \mathcal{O}(\vec{a}^2)$$

$$\mathcal{L} \to \mathcal{L} + \vec{a} \cdot \nabla\mathcal{L} = \mathcal{L} + a^i \partial_j (\delta_i^j \mathcal{L})$$

$$\frac{\partial \phi}{\partial a_i}\Big|_{a_i = 0} = \partial^i \phi$$

Then by Noether's theorem,

$$P^i = \int d^3x \ \Pi(\vec{x}, t)\, \partial^i \phi(\vec{x}, t)$$

$$= -\int d^3x \ \Pi(\vec{x}, t)\, \nabla_i \phi(\vec{x}, t)$$

Recall the plane wave decomposition of ϕ and π:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right)$$

$$\pi(x) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}}\, i \left(-a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right)$$

We also have,

$$\nabla \phi = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}}\, i\vec{k} \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right)$$

Hence,
$$\vec{P} = \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^6} \frac{1}{2} \vec{k}' \left(a_k e^{-ik\cdot x} - a_k^\dagger e^{ik\cdot x} \right) \times \left(-a_{k'} e^{-ik'\cdot x} + a_{k'}^\dagger e^{ik'\cdot x} \right)$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6} \cdot \frac{\vec{k}'}{2} (2\pi)^3 \Big(-a_k a_{k'} \delta^3(\vec{k}+\vec{k}') e^{-it(\omega_k+\omega_{k'})} - a_k^\dagger a_{k'}^\dagger \delta^3(\vec{k}+\vec{k}') e^{it(\omega_k+\omega_{k'})} + a_k a_{k'}^\dagger \delta^3(\vec{k}-\vec{k}') e^{-it(\omega_k-\omega_{k'})} + a_k^\dagger a_{k'} \delta^3(\vec{k}-\vec{k}') e^{it(\omega_k-\omega_{k'})} \Big)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \vec{k} \left(a_{\vec{k}} a_{-\vec{k}} e^{-2i\omega_k t} + a_{\vec{k}}^\dagger a_{-\vec{k}}^\dagger e^{2i\omega_k t} + a_{\vec{k}} a_{\vec{k}}^\dagger + a_{\vec{k}}^\dagger a_{\vec{k}} \right)$$

The first two terms vanish because they are odd under $\vec{k} \to -\vec{k}$.
Combining the last two terms using the harmonic oscillator commutation relations,

$$\vec{P} = \int \frac{d^3k}{(2\pi)^3} \vec{k} \, a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \vec{k} \, \delta^3(\vec{0}) (2\pi)^3$$

We got another $\delta^3(\vec{0})$, but we already learned not to be too worried. This contribution to the momentum is even less worrisome than the corresponding term in the Hamiltonian. If we think of the delta fn as a divergent constant, the integral over $\vec{k}$ is odd, so integrates to zero, anyway.
Hence,

$$\boxed{\vec{P} = :\vec{P}: = \int \frac{d^3k}{(2\pi)^3} \vec{k} \, a^\dagger_{\vec{k}} a_{\vec{k}}}$$

The state $a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle$ is an eigenstate of $\vec{P}$ with eigenvalue $\vec{k}_1 + \cdots + \vec{k}_n$.

The same state is an eigenstate of H with eigenvalue $\omega_{\vec{k}_1} + \cdots + \omega_{\vec{k}_n}$.

Hence, we interpret $a^\dagger_{\vec{k}}$ as a creation operator for a particle with momentum $\vec{k}$ and energy $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$.

Normalization of States

We will normalize states in such a way that the inner product is Lorentz invariant.

The vacuum is normalized as usual: $\boxed{\langle 0|0\rangle = 1.}$

The 1-particle state with momentum $\vec{k}$ we will call $\boxed{|\vec{k}\rangle = \sqrt{2\omega_{\vec{k}}}\; a^{\dagger}_{\vec{k}}|0\rangle}$.

Then $$\boxed{\langle \vec{k}_1|\vec{k}_2\rangle = 2\omega_{\vec{k}_1}(2\pi)^3\,\delta^3(\vec{k}_1-\vec{k}_2)}$$

To see that the inner product is Lorentz invariant, recall that the measure $\frac{d^3k}{2\omega_{\vec{k}}}$ is Lorentz invariant.

Also, $\int \frac{d^3k}{2\omega_{\vec{k}}}\cdot 2\omega_{\vec{k}}\,\delta^3(\vec{k}-\vec{k}') = 1$ is Lorentz inv.

Hence, the integrand $2\omega_{\vec{k}}\,\delta^3(\vec{k}-\vec{k}')$ is Lorentz inv. as well.

<u>Exercise</u>: This can be checked explicitly by letting $k^\mu \to \Lambda^\mu{}_\nu k^\nu$ and using $\delta[f(k)-f(\bar{k})] = \frac{1}{|f'(\bar{k})|}\delta(k-\bar{k})$

The completeness relation for 1-particle states is then,

$$\boxed{\mathbb{1}_{\text{1-particle}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} |\vec{k}\rangle\langle\vec{k}|}$$

<u>Check</u>: $\langle \vec{k}_1 | \vec{k}_2 \rangle \stackrel{?}{=} \langle \vec{k}_1 | \mathbb{1}_{\text{1-particle}} | \vec{k}_2 \rangle$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \cdot 2\omega_{k_1} \delta^3(\vec{k}_1 - \vec{k}) \cdot 2\omega_{k_2} \delta^3(\vec{k} - \vec{k}_2)(2\pi)^6$$

$$= (2\pi)^3 2\omega_{k_1} \delta^3(\vec{k}_1 - \vec{k}_2) \quad \checkmark$$

What is the state $\phi(\vec{x}, t)|0\rangle$?

Consider $\langle \vec{k} | \phi(\vec{x}, t) | 0 \rangle$

$$= \langle 0 | \sqrt{2\omega_k}\, a_k \int \frac{d^3k'}{(2\pi)^3 \sqrt{2\omega_{k'}}} \left(a_{k'} e^{-ik'\cdot x} + a^\dagger_{k'} e^{ik'\cdot x} \right) |0\rangle$$

$$= \langle 0 | \int \frac{d^3k'}{(2\pi)^3} \sqrt{\frac{\omega_k}{\omega_{k'}}} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') e^{ik'\cdot x} | 0 \rangle$$

$$= e^{ik\cdot x}$$

This looks a lot like $\langle \vec{p} | \vec{x} \rangle = e^{-i\vec{p}\cdot\vec{x}}$ in nonrelativistic QM. Hence we are led to interpret $\phi(\vec{x}, t)|0\rangle$ as the state of one particle at position $\vec{x}$ at time t.

Comparison of Quantum Field Theory to relativistic QM:

Recall that one of the reasons we gave up on the Klein-Gordon eqn as a relativistic Schrodinger eqn for a single particle wavefunction was the existence of negative energy states.

The positive energy solutions are $e^{-ik\cdot x}$, which multiply a_k.
The negative energy solutions are $e^{ik\cdot x}$, which multiply $a_k^\dagger$.

But a_k annihilates particles w/ energy $+\omega_{\vec{k}}$, and $a_k^\dagger$ creates particles w/ energy $+\omega_{\vec{k}}$. There are no negative energy states in QFT.

The other reason we gave up on the Klein-Gordon equation in relativistic QM was that the conserved quantity did not have the form of a probability.
So what is it in QFT?

To answer this we have to consider complex scalar fields because there is no analogous conserved quantity for real scalar fields. (Recall that we assumed in our discussion of relativistic QM that the wavefunction was complex. You can check that the conserved current vanishes if ϕ is real.)

Free Complex Scalar Fields

Complex scalars are a simple generalization of real scalars.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{-ik\cdot x} + b_{\vec{k}}^\dagger e^{ik\cdot x} \right)$$

$a_{\vec{k}}$ and $b_{\vec{k}}^\dagger$ are not related anymore because $\phi(x)$ is not Hermitian.

Treating $\phi(x)$ and $\phi(x)^\dagger$ as independent, the canonical commutation relations are:

Exercise:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = [\phi^\dagger(\vec{x},t), \phi^\dagger(\vec{y},t)] = [\phi(\vec{x},t), \phi^\dagger(\vec{y},t)] = 0$$

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = [\phi^\dagger(\vec{x},t), \dot{\phi}^\dagger(\vec{y},t)] = 0$$

$$[\phi(\vec{x},t), \dot{\phi}^\dagger(\vec{y},t)] = [\phi^\dagger(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$$

Everything goes through as for real scalars, but with two kinds of particles: $a_{\vec{k}}^\dagger|0\rangle \neq b_{\vec{k}}^\dagger|0\rangle$.

Exercise:

$$[a_k, a_{k'}] = [b_k, b_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = [b_k^\dagger, b_{k'}^\dagger] = 0$$

$$[a_k, b_{k'}] = [a_k^\dagger, b_{k'}^\dagger] = [a_k, b_{k'}^\dagger] = [a_k^\dagger, b_{k'}] = 0$$

$$[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$$

What about the conserved charge?

The symmetry is $\phi(x) \to e^{-i\theta}\phi(x)$, $\phi^\dagger(x) \to e^{i\theta}\phi^\dagger(x)$.

$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2$

$$J^\mu = i\left(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right)$$

The conserved charge is $Q = \int d^3x\, J^0$

$$Q = \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}}\, i\Big((a_k^\dagger e^{ik\cdot x} + b_k e^{-ik\cdot x}) \cdot i\omega_{k'} \times (-a_{k'} e^{-ik'\cdot x} + b_{k'}^\dagger e^{ik'\cdot x})$$

$$- i\omega_k (a_k^\dagger e^{ik\cdot x} - b_k e^{-ik\cdot x})(a_{k'} e^{-ik'\cdot x} + b_{k'}^\dagger e^{ik'\cdot x})\Big)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}(-\omega_k)\Big(-a_k^\dagger a_k + b_k b_k^\dagger - b_k a_{-k} e^{-2i\omega_k t} + a_k^\dagger b_{-k}^\dagger e^{2i\omega_k t}\Big) + h.c.$$

$$= \int \frac{d^3k}{(2\pi)^3} \left(a_k^\dagger a_k - b_k^\dagger b_k\right) + \text{const.}$$

As usual, we can eliminate the constant by normal ordering, redefining $Q \to :Q:$

So $a_k^\dagger$ creates particles w/ positive charge.
$b_k^\dagger$ creates particles w/ negative charge.

$$[Q, a_{k'}] = \int \frac{d^3k}{(2\pi)^3} [a_k^\dagger, a_{k'}] a_k = -a_{k'}$$

$$[Q, a_{k'}^\dagger] = a_{k'}^\dagger$$

$$[Q, b_{k'}] = b_{k'}$$

$$[Q, b_{k'}^\dagger] = -b_{k'}^\dagger$$

The particle created by $b_k^\dagger$ has the same energy and momentum as the state created by $a_k^\dagger$, but they carry opposite charge.

The particle created by $b_k^\dagger$ is called the <u>antiparticle</u> of the particle created by $a_k^\dagger$.

- Since $b_k^\dagger$ multiplies $e^{ik\cdot x}$ in $\phi(x)$, we see that what used to be a negative energy state has become a positive energy antiparticle state.
- We will see that the existence of antiparticles is crucial for <u>causality</u> in QFT.
- Real fields create and annihilate the same type of particle, so for real fields the particles are their own antiparticles.