Cross Sections and Decay Rates

The amplitude for an initial state of particles with momenta p_i to evolve into a final state of particles with momenta p_f is $\propto \delta^4(\Sigma p_i - \Sigma p_f)$. Squaring the amplitude gives a factor of $(\delta^4(\Sigma p_i - \Sigma p_f))^2$. We need to make sense of this.

In a real experiment each particle is described by a normalizable wavepacket that is localized far from the other particles at early and late times.

A 1-particle state with wavefunction $\varphi(\vec{k})$ is:

$$\boxed{|\varphi(\vec{k})\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\, \varphi(\vec{k})\, |\vec{k}\rangle}$$

(suppressing quantum numbers other than momentum.)

The wavefunctions are normalized as follows:

$$1 = \langle\varphi|\varphi\rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k\, 2\omega_{k'}}}\, \varphi(\vec{k})\varphi(\vec{k}')^* \langle\vec{k}'|\vec{k}\rangle$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k\, 2\omega_{k'}}}\, \varphi(\vec{k})\varphi(\vec{k}')^* \sqrt{2\omega_k}\sqrt{2\omega_k}\, \langle 0|a_{\vec{k}'}\, a^\dagger_{\vec{k}}|0\rangle$$

$$= \boxed{\int \frac{d^3k}{(2\pi)^3}\, |\varphi(\vec{k})|^2 = 1}$$

We are typically interested in one of two physical situations:

1) A beam of some type of particle collides with a target, which can be fixed or could be another beam of particles. We want to know the transition rate for scattering into some region of final states, per unit time, per unit flux of particles:
This defines the <u>cross section</u>,

$$\boxed{\sigma = \frac{\#\text{ events}}{\text{unit time} \times \text{unit flux}}}$$

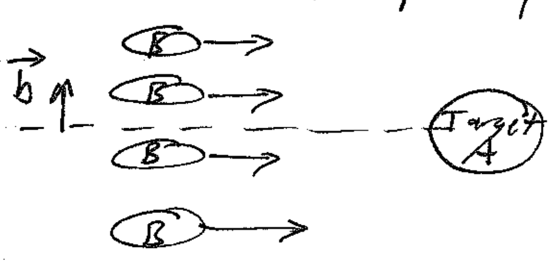
The flux is the # particles/(cross-sectional area × unit time), so σ has the dimensions of area.

2) A particle, or a collection of particles, decay into lighter particles. We want to know the decay rate per unit time,

$$\boxed{\Gamma = \frac{\#\text{ decays}}{\text{unit time}}}$$

We will consider these two situations separately.

1) <u>Cross Sections</u>: Scatter incident wavepackets uniformly distributed in impact parameter $\vec{b}$



Recall that the momentum operator generates translations, so we include a factor of $e^{-i\vec{b}\cdot\vec{k}}$ to translate the incident B wavepackets by $\vec{b}$ from the collinear wavepacket w/ $\vec{b}=0$.

$$|\varphi_A \varphi_B\rangle_{in} = \int \frac{d^3k_A\, d^3k_B}{(2\pi)^6 \sqrt{2\omega_{k_A} 2\omega_{k_B}}} \varphi_A(\vec{k}_A)\varphi_B(\vec{k}_B)\, e^{-i\vec{b}\cdot\vec{k}_B}\, |\vec{k}_A\, \vec{k}_B\rangle_{in}$$

We can form wavepackets to describe the out states, but if detectors mainly measure momentum and do not resolve the positions of the final state particles (at least at the level of the de Broglie wavelengths) then it is okay to use out states of definite momenta. (It also makes things a little easier.)

With our relativistically normalized states, the probability for $|\varphi_A\varphi_B\rangle$ to scatter into n particles with momenta in a region $d^3p_1 \cdots d^3p_n$ is

$$P(AB \to 1,2,\dots,n) = \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_{p_f}} \left| {}_{out}\langle p_1 \cdots p_n | \varphi_A \varphi_B\rangle_{in} \right|^2$$

For a single target A and many incident particles B with n_B incident particles/unit area evenly distributed in impact parameters $\vec{b}$, the # scattering events is

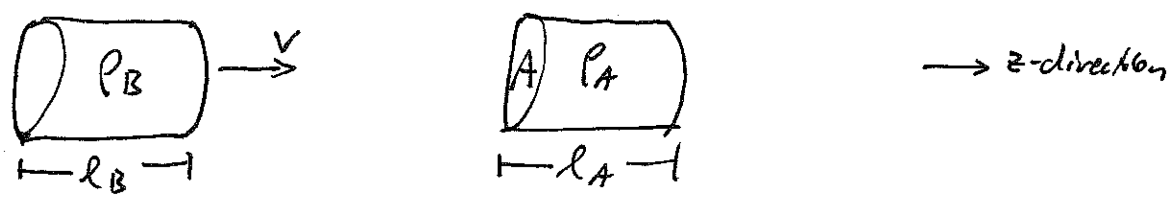
$$N = \sum_{\substack{\text{incident} \\ \text{particles } i}} P_i = \int d^2b\; n_B\, P(\vec{b}).$$

The # of incident particles / unit area is
n_B = flux × time.

The <u>cross section</u> is then $\sigma = \frac{N}{n_B} = \int d^2b\, P(\vec{b})$.

If there are N_A target particles then the number of events grows with N_A. The cross section is defined to be invariant w/r/t N_A.

For example, consider two bunches of particles from a beam, of length ℓ_B and ℓ_A, and densities ρ_B and ρ_A, cross-sectional area A



ρ = # particles/unit volume

$$n_B = \rho_B \ell_B \quad , \quad n_A = \rho_A \ell_A$$
$$N_A = \rho_A \ell_A \cdot A$$

Then $\sigma = \frac{\#\text{ events}}{n_B N_A}$.

The <u>differential cross section</u> for scattering into a region of momenta $\prod_f d^3p_f$ is:

$$d\sigma = \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_{p_f}} \int d^2b \left| {}_{out}\langle p_1 \cdots p_n | \varphi_A\, \varphi_B(\vec{b}) \rangle_{in} \right|^2$$

In terms of the scattering matrix S,

$$d\sigma = \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_{p_f}} \int d^2 b \int \frac{d^3 k_A \, d^3 k_B}{(2\pi)^6 \sqrt{2\omega_{k_A}} \sqrt{2\omega_{k_B}}} \int \frac{d^3 k_A' \, d^3 k_B'}{(2\pi)^6 \sqrt{2\omega_{k_A'}} \sqrt{2\omega_{k_B'}}}$$

$$\varphi_A(k_A)\, \varphi_B(k_B)\, \varphi_A^*(k_A')\, \varphi_B^*(k_B')\, e^{-i\vec{b}\cdot(\vec{k}_B - \vec{k}_B')}$$

$$\langle p_1 \cdots p_n | (S-1) | k_A k_B \rangle \langle p_1 \cdots p_n | (S-1) | k_A' k_B' \rangle^*$$

Define the <u>invariant scattering amplitude</u> $\mathcal{M}(k_A k_B \to p_1 \cdots p_n)$ by factoring out the momentum-conserving δ-fn from the S-matrix element:

$$\langle p_1 \cdots p_n | (S-1) | k_A k_B \rangle \equiv i\mathcal{M}(k_A k_B \to p_1 \cdots p_n)\, (2\pi)^4 \delta^4(k_A + k_B - \Sigma p_f)$$

Also use $\int d^2 b \, e^{-i\vec{b}\cdot(\vec{k}_B - \vec{k}_B')} = (2\pi)^2 \delta^2(k_B^\perp - k_B'^\perp)$

Consider the factor,

$$\int \frac{d^3 k_A' \, d^3 k_B'}{(2\pi)^6 \sqrt{2\omega_A' 2\omega_B'}} \varphi_A^*(k_A') \varphi_B^*(k_B') \left(-i\mathcal{M}(k_A k_B \to p_1 \cdots p_n)^*\right) (2\pi)^6 \delta^4(k_A' + k_B' - \Sigma p_f) \times \delta^2(k_B^\perp - k_B'^\perp)$$

$$= \int \frac{d^3 k_A' \, dk_B'^z}{\sqrt{2\omega_A' 2\omega_B'}} \varphi_A^*(k_A') \varphi_B^*(k_B') (-i\mathcal{M}^*) \delta^2(k_A'^\perp + k_B^\perp - \Sigma p_f^\perp) \times \delta(k_A'^z + k_B'^z - \Sigma p_f^z) \delta(\omega_A' + \omega_B' - \Sigma \omega_f)$$

$$= \int \frac{dk_A'^z \, dk_B'^z}{\sqrt{2\omega_A' 2\omega_B'}} \varphi_A^*(k_A'^z, \vec{k}_A^\perp) \varphi_B^*(k_B'^z, \vec{k}_B^\perp) (-i\mathcal{M}^*) \delta(k_A'^z + k_B'^z - \Sigma p_f^z) \times \delta(\omega_A' + \omega_B' - \Sigma \omega_f)$$

$$= \int \frac{dk_A'^z}{\sqrt{2\omega_A' 2\omega_B}} \varphi_A^*(k_A'^z, k_A^\perp)\varphi_B^*(\vec{k}_B)(-iM^*)\,\delta\left(\sqrt{k_A'^2+m_A^2}+\sqrt{k_B'^2+m_B^2}-\sum\omega_f\right)\Big|_{k_B'^z = \sum p_f^z - k_A'^z}$$

$$= \frac{1}{\sqrt{2\omega_A 2\omega_B}} \varphi_A^*(\vec{k}_A)\varphi_B^*(\vec{k}_B)(-iM^*)\frac{1}{\left|\frac{k_A'^z}{\omega_A'} - \frac{k_B'^z}{\omega_B'}\right|}\Bigg|_{\substack{k_A'^z + k_B'^z = \sum p_f^z \\ \omega_A' + \omega_B' = \sum \omega_f}}$$

$$\frac{\vec{k}}{\omega} = \vec{v} \implies \frac{1}{\left|\frac{k_A'^z}{\omega_A'} - \frac{k_B'^z}{\omega_B'}\right|} = \frac{1}{|\vec{v}_A - \vec{v}_B|}$$

$|\vec{v}_A - \vec{v}_B|$ = relative velocity in lab frame.

Putting it together,

$$d\sigma = \prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\omega_f} \frac{|M(p_A, p_B \to p_1\, p_2 \cdots p_n)|^2}{2\omega_A 2\omega_B |v_A - v_B|}$$

$$\int \frac{d^3k_A\, d^3k_B}{(2\pi)^6} |\varphi_A(k_A)|^2 |\varphi_B(k_B)|^2 (2\pi)^4 \delta^4(k_A + k_B - \sum p_f)$$

We have assumed that the wavepackets are localized around $\vec{p}_A$ and $\vec{p}_B$.
If the detecter cannot resolve the spread in momentum due to the fact that $\varphi_A(k)$ and $\varphi_B(k)$ are not δ-fns, then we can replace (k_A, k_B) by (p_A, p_B) in the $\delta^4(k_A + k_B - \sum p_f)$.

We then obtain our final expression for the differential cross section:

$$d\sigma = \frac{|M(P_A,P_B \to P_1 \cdots P_n)|^2}{2\omega_A\, 2\omega_B\, |V_A - V_B|} \underbrace{\left(\prod_{f=1}^{n} \frac{d^3P_f}{(2\pi)^3 2\omega_f}\right)(2\pi)^4 \delta^4\left(P_A + P_B - \Sigma P_f\right)}_{D_n \,\equiv\, n\text{-body invariant phase space density}}$$

Note that $\omega_A \omega_B |\vec{V}_A - \vec{V}_B| = |\omega_B P_A^z - \omega_A P_B^z| = |\epsilon_{\mu x y \nu} P_A^\mu P_B^\nu|$ is Lorentz invariant under boosts in the z-direction. Hence it is the same in the lab frame and the COM frame.

D_2: 2-body final state in COM frame

$$D_2 = \frac{d^3P_1\, d^3P_2}{(2\pi)^6\, 2\omega_1\, 2\omega_2} \quad (2\pi)^4 \delta^3(\vec{P}_1 + \vec{P}_2)\, \delta(\omega_1 + \omega_2 - \omega_{Tot})$$

$$= \frac{d^3P_1}{(2\pi)^3 \cdot 4\omega_1\omega_2}\; 2\pi\, \delta(\omega_1 + \omega_2 - \omega_{Tot})\Big|_{\vec{P}_2 = -\vec{P}_1}$$

Use $\omega_1^2 = \vec{P}_1^{\,2} + m_1^2$, $\omega_2^2 = \vec{P}_2^{\,2} + m_2^2 = \vec{P}_1^{\,2} + m_2^2$

Define $P_1 \equiv |\vec{P}_1| \rightarrow \delta(\omega_1 + \omega_2 - \omega_{Tot}) = \left.\dfrac{\delta(P_1 - \bar{P}_1)}{\frac{\partial(\omega_1+\omega_2)}{\partial P_1}}\right|_{\bar{P}_1 \text{ satisfies } \omega_1+\omega_2 = \omega_{Tot}}$

$$\frac{\partial(\omega_1+\omega_2)}{\partial P_1} = \frac{P_1}{\omega_1} + \frac{P_1}{\omega_2} = \frac{P_1(\omega_1+\omega_2)}{\omega_1\omega_2} = \frac{P_1\,\omega_{Tot}}{\omega_1\omega_2}$$

Also use $d^3P_1 = dP_1\, d\Omega_1$. Then,

$$\boxed{D_2 = \frac{1}{16\pi^2}\,\frac{P_1\, d\Omega_1}{\omega_{Tot}}}$$

The differential cross section w/ 2-body final state is then,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\omega_A \omega_B} |M|^2 \frac{1}{|\vec{v}_A - \vec{v}_B|} \cdot \frac{p_1}{16\pi^2 \omega_{Tot}}$$

In COM frame: $\omega_A \omega_B |\vec{v}_A - \vec{v}_B| = |\omega_B \vec{p}_A - \omega_A \vec{p}_B|$

$$= |\omega_B \vec{p}_A + \omega_A \vec{p}_A|$$
$$= \omega_{Tot}\, p_A$$

So,

final momentum $p_1 = |\vec{p}_1|$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \omega_{Tot}^2} \frac{p_1}{p_A} |M|^2}$$

initial momentum $p_A = |\vec{p}_A|$

Note that $\frac{d\sigma}{d\Omega} \to \infty$ if $p_A \to 0$ and $p_1 \neq 0$.

As $p_A \to 0$, the amount of time the two particles remain near one another grows, so there's more time for scattering to occur.

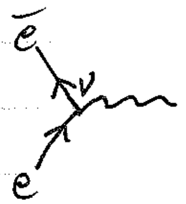
<u>Example</u>: $e^+e^- \to \mu^+\mu^-$

The Lagrangian for QED coupled to electrons and muons is,

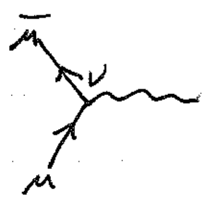
$$\mathcal{L} = \bar{\psi}_e (i\not{\partial} - m_e - e\not{A})\psi_e + \bar{\psi}_\mu (i\not{\partial} - m_\mu - e\not{A})\psi_\mu - \frac{1}{4} F_{\mu\nu}^2,$$

where ψ_e is the electron field, and ψ_μ is the muon field

There are two vertices now:



$(-ie\gamma^\nu)$



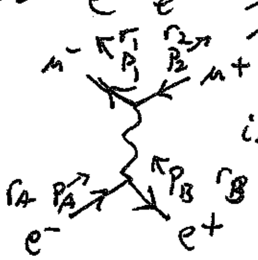
$(-ie\gamma^\nu)$

And two fermion propagators:

$e \xrightarrow{k \to}$ $\dfrac{i(\not{k}+m_e)}{k^2-m_e^2+i\epsilon}$

$\mu \xrightarrow{k \to}$ $\dfrac{i(\not{k}+m_\mu)}{k^2-m_\mu^2+i\epsilon}$

Because the electron and muon are distinguishable, there is only one Feynman diagram that contributes to $e^+e^- \to \mu^+\mu^-$:



$$iM = \bar{u}^{r_1}(p_1)(-ie\gamma^\mu)v^{r_2}(p_2)\ \bar{v}^{r_B}(p_B)(-ie\gamma^\nu)u^{r_A}(p_A) \times \frac{(-ig_{\mu\nu})}{(p_A+p_B)^2}$$

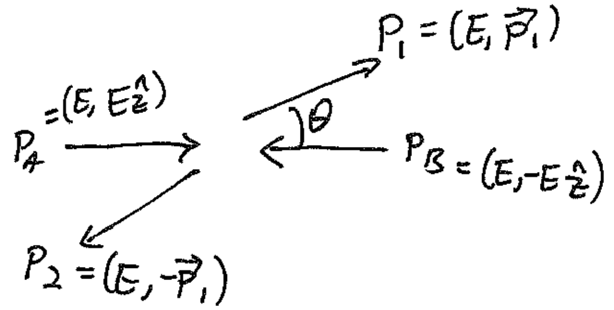
where we have factored out the $(2\pi)^4\delta^4(p_A+p_B-p_1-p_2)$.

Squaring the amplitude and avging over initial spins, summing over final spins gives,

$$\frac{1}{4}\sum_{\text{spins}}|M|^2 \overset{\text{Exercise}}{=\!=} \frac{8e^4}{(p_A+p_B)^4}\Big[(p_A\cdot p_1)(p_B\cdot p_2)+(p_A\cdot p_2)(p_B\cdot p_1) + m_\mu^2(p_A\cdot p_B)\Big]$$

where we have approximated $m_e=0$ because $\frac{m_e}{m_\mu}\sim\frac{1}{200}$.

Kinematics: Center-of-Mass frame



$$|\vec{P}_1| = \sqrt{E^2 - m_\mu^2}$$
$$\vec{P}_1 \cdot \hat{z} = |\vec{P}_1| \cos\theta$$

To compute the differential cross section we need to express kinematic factors in terms of E and θ:

$$(P_A + P_B)^2 = 4E^2, \qquad (P_A \cdot P_B) = 2E^2, \quad E_{Tot} = 2E$$
$$(P_A \cdot P_1) = (P_B \cdot P_2) = E^2 - E|\vec{P}_1|\cos\theta$$
$$(P_A \cdot P_2) = (P_B \cdot P_1) = E^2 + E|\vec{P}_1|\cos\theta$$

Then
$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{8e^4}{16E^4}\left[E^2(E - |\vec{P}_1|\cos\theta)^2 + E^2(E + |\vec{P}_1|\cos\theta)^2 + 2m_\mu^2 E^2\right]$$
$$= \frac{e^4}{2E^4}\left[2(E^4 + m_\mu^2 E^2) + 2|\vec{P}_1|^2\cos^2\theta\right]$$
$$= e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

Since we are treating the electrons as massless, we also have $|\vec{v}_A - \vec{v}_B| = 2$. Putting it together,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(2E)^2}\frac{\sqrt{E^2 - m_\mu^2}}{E}e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$\left(\alpha = \frac{e^2}{4\pi}\right)$
$$= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

Integrating over the <u>scattering angle</u> θ gives the total X-section:

$$\sigma_{Tot} = \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{\alpha^2}{16E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)\cdot 4\pi + \frac{4\pi}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right]$$

$$= \frac{4\pi\alpha^2}{48E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[4+2\frac{m_\mu^2}{E^2}\right]$$

There is a lot more to say about the $e^+e^- \to \mu^+\mu^-$ cross section, but we're short on time so we'll move on.

2) <u>Decay Rates</u>: Consider a 1-particle wavepacket centered about the rest frame $P^\mu_{(0)} = (m, \vec{0})$.

$$|\varphi\rangle_{in} = \int \frac{d^3K}{(2\pi)^3\sqrt{2\omega_K}}\, \varphi(\vec{K})\, |\vec{K}\rangle_{in}, \text{ where}$$

$$\int \frac{d^3K}{(2\pi)^3} |\varphi(\vec{K})|^2 = 1, \quad {}_{in}\langle\varphi|\varphi\rangle_{in} = 1.$$

The probability of decaying into a final state w/ particles around momenta $p_1, p_2, \dots, p_n$ is

$$P(\varphi \to 1,2,\dots,n) = \prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_{p_f}} \left|{}_{out}\langle p_1\cdots p_n|\varphi\rangle_{in}\right|^2$$

$$P(\varphi \to 1 \cdots n) = \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \left| \langle p_1 \cdots p_n | (S-1) | \varphi \rangle \right|^2$$

$$= \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \int \frac{d^3 k \, d^3 k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \varphi(\vec{k}) \varphi(\vec{k}')^* \langle p_1 \cdots p_n | (S-1) | k \rangle$$

$$\times \langle p_1 \cdots p_n | (S-1) | k' \rangle^*$$

$$= \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \int \frac{d^3 k \, d^3 k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \varphi(k) \varphi(k')^* \, i\mathcal{M}(k \to p_1 \cdots p_n) (2\pi)^4 \delta^4(k - \Sigma p_f)$$

$$\times \left(-i\mathcal{M}(k' \to p_1 \cdots p_n)^*\right) (2\pi)^4 \delta^4(k' - \Sigma p_f)$$

$$\approx \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \frac{|\mathcal{M}(p_{(0)} \to p_1 \cdots p_n)|^2}{2\omega_0} (2\pi)^4 \delta^4(p_{(0)} - \Sigma p_f)$$

$$\times (2\pi) \delta(\omega_0 - \Sigma \omega_f) \int \frac{d^3 k}{(2\pi)^3} |\varphi(\vec{k})|^2$$

$$= \prod_f \frac{d^3 p_f}{(2\pi)^3 2\omega_f} \frac{|\mathcal{M}|^2}{2m} (2\pi)^3 \delta^3(\vec{p}_0 - \Sigma \vec{p}_f) (2\pi)^2 \left(\delta(\omega_0 - \Sigma \omega_f)\right)^2.$$

We got another δ-fn squared. If a particle is unstable and decays we can't really make an in-state in the infinite past. But it does make sense to consider the probability of decay <u>per unit time</u>.

We can write $(2\pi)^2 (\delta(\omega_0 - \Sigma\omega_f))^2 = \int dt\, dt' \, e^{i(\omega_0 - \Sigma\omega_f)(t+t')}$

$$= (2\pi) \delta(\omega_0 - \Sigma\omega_f) \int dt$$
$$= (2\pi T) \delta(\omega_0 - \Sigma\omega_f)$$

We now divide by this factor of T.

The differential decay rate is:

$$\boxed{d\Gamma = \frac{P(a \to p_1 \cdots p_n)}{T} = \frac{1}{2m} \prod_{f=1}^{n} \frac{d^3 p_f}{(2\pi)^3 2\omega_f} (2\pi)^4 \delta^4(p_{(0)} - \Sigma p_f) \times |M(a \to p_1 \cdots p_n)|^2}$$

In terms of the n-body invariant phase space factor D_n,

$$d\Gamma = \frac{1}{2m} |M|^2 D_n .$$

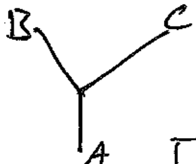
The total decay rate is $\Gamma = \int d\Gamma$.

<u>Example</u>: Three real scalars A, B, C with coupling
$\mathcal{L}_I = -g ABC$.

Assume $m_A > m_B + m_C$ so that the decay $A \to B + C$ is kinematically allowed.

The lowest order invariant amplitude is

$p_A = (m, \vec{0})$
$p_B = (\omega_B, \vec{p})$
$p_C = (\omega_C, -\vec{p})$
$\omega_{-} = m_A$



$iM = -ig$. Using our previous expression for D_2,

$$\boxed{d\Gamma = \frac{1}{2m_A} \cdot g^2 \cdot \frac{1}{16\pi^2} \frac{|\vec{p}|\, d\Omega}{m_A}},$$

where $|\vec{p}|$ is determined by $m_A = \sqrt{|\vec{p}|^2 + m_B^2} + \sqrt{|\vec{p}|^2 + m_C^2}$

Total width:

$$\Gamma = \int d\Gamma = \frac{g^2 |\vec{p}|}{8\pi\, m_A^2}$$