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Wick's Theorem

Consider QED: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

$$\mathcal{L}_I = -e\,\bar{\Psi} A\!\!\!/ \Psi$$

$\mathcal{L}_0$ is the Lagrangian for a free vector field and a free Dirac field.

$\mathcal{L}_I$ describes the interaction between the otherwise free fields.

If $\mathcal{L}_I$ does not contain derivatives (as in QED) then the canonical momenta for the fields are determined by the free Lagrangian $\mathcal{L}_0$:

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \frac{\partial \mathcal{L}_0}{\partial(\partial_0 \psi)} \quad , \text{etc.}$$

Then the Hamiltonian decomposes into a free part H_0 determined by $\mathcal{L}_0$, and an interacting part H_I, with $\boxed{H_I = -\int d^3x\, \mathcal{L}_I}$.

Expanding $U_I(t,t')$ in H_I will give, for example at second order in e,

$$U_I(t,t') \supset \frac{(-i)^2 e^2}{2!} \int_{t'}^{t} dt_1 \int d^3x_1 \int_{t'}^{t} dt_2 \int d^3x_2 \; T\left[\bar{\Psi}\not{A}\Psi(x_1)\, \bar{\Psi}\not{A}\Psi(x_2)\right]$$

The fields appearing in the expansion of $U_I(t,t')$ are free fields, and can be expanded in creation and annihilation operators.

In scattering theory we will want to calculate matrix elements between momentum states, which will involve integrals of matrix elements of time-ordered products of fields, for example,

$$\langle \vec{k}_3, \vec{k}_4 |\, T[\bar{\Psi}\not{A}\Psi(x_1)\, \bar{\Psi}\not{A}\Psi(x_2)]\, | \vec{k}_1, \vec{k}_2 \rangle ,$$

where $|\vec{k}_1, \vec{k}_2\rangle$ is a two-electron momentum eigenstate.

This would be a lot easier to calculate if we could move all the a_k's and $a_k^\dagger$'s in the expansion of the free fields into a normal-ordered form.

<u>Wick's theorem</u> provides the bookkeeping for reordering time-ordered products into normal-ordered products.

Define the <u>contraction</u> of two free fields A and B:

$$\boxed{\overbracket{A(x)B(y)} \equiv T\big(A(x)B(y)\big) - :A(x)\,B(y):}$$

For now assume that A and B are bosonic (scalars or vectors).

Decompose the free fields into the part with creation operators $A^{(-)}$ and the part with annihilation operators $A^{(+)}$.

Suppose $x^0 > y^0$.

$$\begin{aligned} T\big(A(x)\,B(y)\big) &= A(x)\,B(y) \\ &= \left(A^{(+)} + A^{(-)}\right)\left(B^{(+)} + B^{(-)}\right) \\ &= \left(A^{(+)}B^{(+)} + A^{(-)}B^{(-)} + A^{(-)}B^{(+)} + B^{(-)}A^{(+)}\right) + \left[A^{(+)}, B^{(-)}\right] \\ &= :A(x)\,B(y): + \left[A^{(+)}(x), B^{(-)}(y)\right] \end{aligned}$$

The contraction is $\boxed{\overbracket{A(x)\,B(y)} = \left[A^{(+)}(x),\, B^{(-)}(y)\right]}$

From the commutators between creation and annihilation ops, the contraction is a c-number function of x, y. Hence it equals its vacuum expectation value (in the free field vacuum). Similarly when $y^0 > x^0$.

Hence,

$$\begin{aligned} \overbracket{A(x)\,B(y)} &= \langle 0|\, \overbracket{A(x)B(y)}\, |0\rangle \\ &= \langle 0|\, T\big(A(x)\,B(y)\big)|0\rangle - \langle 0|\, :\cancelto{0}{A(x)\,B(y)}:\, |0\rangle \end{aligned}$$

So we have learned that: $\boxed{\overbracket{A(x)\,B(y)} = \langle 0|\, T(A(x)\,B(y))\,|0\rangle}$.

If A and B are real scalar fields then

$$\overbracket{\phi(x)\,\phi(y)} = \langle 0|\, T(\phi(x)\,\phi(y))\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

For complex scalars:

$$\overbracket{\phi(x)\,\phi^{+}(y)} = \int \frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

$$\overbracket{\phi(x)\,\phi(y)} = \overbracket{\phi^{+}(x)\,\phi^{+}(y)} = 0$$

For massless vector field:

$$\overbracket{A_\mu(x)\,A_\nu(y)} = \int \frac{d^4k}{(2\pi)^4}\,\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\,e^{-ik\cdot(x-y)}$$

Contractions in normal-ordered products (of bosonic fields):

$$:A(x)\,\overbrace{B(y)\,C(z)\,D(w)}:\;\equiv\;:A(x)\,C(z):\,\overbrace{B(y)\,D(w)}$$

Write $\phi_1 \equiv \phi_1(x_1)$, $\phi_2 \equiv \phi_2(x_2)$, etc. as generic bosonic fields at the pts x_1, x_2, etc.

<u>Wick's theorem</u>: $T(\phi_1 \cdots \phi_n) = :\phi_1 \cdots \phi_n:$
$+ :\overbrace{\phi_1 \phi_2}\phi_3 \cdots \phi_n: +$ all other terms w/ one contraction
$+ :\overbrace{\phi_1 \phi_2}\,\overbrace{\phi_3 \phi_4}\,\phi_5 \cdots \phi_n: +$ all other terms w/ two contractions
$+ \cdots$
$+$ either $:\overbrace{\phi_1 \phi_2}\,\overbrace{\phi_3 \phi_4} \cdots \overbrace{\phi_{n-1} \phi_n}:$ if n is even
or $:\overbrace{\phi_1 \phi_2}\,\overbrace{\phi_3 \phi_4} \cdots \overbrace{\phi_{n-2} \phi_{n-1}}\,\phi_n:$ if n is odd

In other words, $T(\phi_1 \cdots \phi_n)$ contains all possible normal-ordered contractions, each with coefficient 1.

<u>Proof</u> (by induction): $n=1$ $\;T(\phi_1) = :\phi_1:$ $\quad$ (trivial)
$n=2$ $\;T(\phi_1 \phi_2) = :\phi_1 \phi_2: + \overbrace{\phi_1 \phi_2}$ $\quad$ (definition)

$n>2$: Suppose $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$.
Define the right hand side of Wick's identity in terms of normal-ordered contractions as $W(\phi_1 \cdots \phi_n)$.

We want to show that $T(\phi_1 \cdots \phi_n) = W(\phi_1 \cdots \phi_n)$.

Suppose this is true for $T(\phi_2 \cdots \phi_n) = W(\phi_2 \cdots \phi_n)$.

$$T(\phi_1 \cdots \phi_n) = \phi_1 T(\phi_2 \cdots \phi_n)$$
$$= \phi_1 W(\phi_2 \cdots \phi_n)$$

$$= \phi_1^{(-)} W(\phi_2 \cdots \phi_n) + W(\phi_2 \cdots \phi_n)\phi_1^{(+)} + [\phi_1^{(+)}, W(\phi_2 \cdots \phi_n)]$$

The first two terms are normal ordered and contain all possible contractions not including ϕ_1.

By expanding the commutator in the last term as a sum of commutators of $\phi_1^{(+)}$ with each uncontracted field in each term in $W(\phi_2 \cdots \phi_n)$, we see that the last term includes all possible contractions in which ϕ_1 is contracted with some field.

Hence, the right-hand side includes all normal-ordered contractions of $\phi_1 \cdots \phi_n$, so it is $W(\phi_1 \cdots \phi_n)$.

Hence, $T(\phi_1 \cdots \phi_n) = W(\phi_1 \cdots \phi_n)$ as desired.

Wick's theorem for fermions

Fermions satisfy anticommutation relations, so we need extra minus signs in Wick's theorem.

Recall the minus signs in switching the order of fermionic operators in time-ordered and normal-ordered products:

$$T\left(A(x)\,B(y)\right) = -T\left(B(y)\,A(x)\right)$$
$$:A(x)\,B(y): = -:B(y)\,A(x):$$

$$:AB: = A^{(+)}B^{(+)} + A^{(-)}B^{(+)} + A^{(-)}B^{(-)} - B^{(-)}A^{(+)}$$

↑ extra -ve sign from switching the order of $A^{(+)}$ and $B^{(-)}$.

Contractions defined as for bosons: $\overbrace{AB} = T(AB) - :AB:$

Assume $x_A^0 > x_B^0$. Then $T(AB) = AB$
$$= A^{(+)}B^{(+)} + A^{(+)}B^{(-)} + A^{(-)}B^{(+)} + A^{(-)}B^{(-)}$$

$$\overbrace{AB} = \{A^{(+)}, B^{(-)}\} \quad = c\# \text{ fn. of } x_A, x_B.$$

Similarly for $x_A^0 < x_B^0$. Then, $\overbrace{AB} = \langle 0|\,T(AB)\,|0\rangle$.

For Wick's theorem we need to define contractions in normal-ordered products w/ minus signs for odd permutations of fermion fields.

$$:\overbrace{A_1 A_2 A_3} A_4: \; = -\overbrace{A_1 A_3} \; :A_2 A_4:$$

Exercise: With these minus signs the proof of Wick's theorem goes through as for bosonic operators. Then,

$$T(A_1 A_2 A_3 A_4) = \; :A_1 A_2 A_3 A_4 + \text{all possible contractions}:$$

$$\overbrace{\psi(x)\,\bar{\psi}}(y) = \langle 0 | T\,\psi(x)\,\bar{\psi}(y) | 0 \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m)\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

$$\overbrace{\psi(x)\,\psi}(y) = \overbrace{\bar{\psi}(x)\,\bar{\psi}}(y) = 0.$$

For products of fields involving both bosons and fermions, include minus signs for fermions as above, not for bosons.

Example: $T\left(\bar{\psi} A_\mu \psi(x_1) \;\; \bar{\psi} A_\nu \psi(x_2)\right)$

$$= \; :\bar{\psi} A_\mu \psi(x_1) \; \bar{\psi} A_\nu \psi(x_2): \; + \; :\bar{\psi} \overbrace{A_\mu \psi \;\; \bar{\psi} A_\nu} \psi:$$

$$+ \; :\bar{\psi} A_\mu \overbrace{\psi \; \bar{\psi}} A_\nu \psi: \; + \; :\overbrace{\bar{\psi} A_\mu \psi \;\; \bar{\psi} A_\nu \psi}:$$

$$+ \; :\bar{\psi} \overbrace{A_\mu \overbrace{\psi \;\; \bar{\psi}} A_\nu} \psi: \; + \; \ldots$$

We're now ready for diagrammatic perturbation theory.