Electrodynamics

To construct the theories of the free scalar field and the Dirac spinor field we found the most general Lorentz invariant action satisfying a few symmetry principles and a few simplifying assumptions. We can do the same for the vector field.

Consider a vector field $A^\mu(x)$. We construct a Lagrangian satisfying:

1) Lorentz scalar
2) No more than two derivatives
3) At most quadratic in the fields
4) Gauge invariance $\quad A^\mu(x) \to A^\mu(x) + \partial^\mu \theta(x) \quad \forall \theta(x)$

The first condition is for Lorentz invariance. The second and third are for simplicity. The last condition is a property of electrodynamics, and is one of its most important and complicating features.

Products of $A^\mu(x)$ without spacetime derivatives are not gauge invariant.

We can form a gauge invariant combination of derivatives of A^μ by antisymmetrizing. Define the field strength tensor $\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.}$

Under a gauge transformation,

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \theta) - \partial_\nu (A_\mu + \partial_\mu \theta)$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$= F_{\mu\nu}.$$

In terms of $F_{\mu\nu}$ we can write a Lagrangian satisfying conditions 1–4:

$$\mathcal{L} = \pm \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Up to the arbitrary factor of $\frac{1}{4}$ this is the most general Lagrangian satisfying the conditions. As you will see, positivity of the Hamiltonian requires that we pick the −ve sign. Hence we obtain the Lagrangian for free electrodynamics:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

In terms of A^μ, $\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -F^{\mu\nu}$$

Euler–Lagrange Eqs: $\boxed{\partial_\mu F^{\mu\nu} = 0}$

From the definition of $F^{\mu\nu}$ there is also an identity, the <u>Bianchi identity</u>: $\boxed{\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0.}$

Solutions to the Source-Free Maxwell Eqs:

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = 0$$

Ansatz: $A^\mu(x) = \overset{\text{"polarization"}}{\epsilon^\mu(k)}\, e^{-ik\cdot x} + \text{c.c.} \quad , k^0 > 0$

$$(-ik_\mu)\left((-ik^\mu)\epsilon^\nu - (-ik^\nu)\epsilon^\mu \right) = 0$$

$$-k^2 \epsilon^\nu + (k\cdot\epsilon) k^\nu = 0$$

4D <u>Transverse Modes</u>: $\boxed{\begin{matrix} \epsilon\cdot k = 0 \\ k^2 = 0 \end{matrix}} \quad , \epsilon \not\propto k$

4D <u>Longitudinal Modes?</u> Assume $\epsilon \propto k$. Then Maxwell's eqs become:

$$-k^2 \epsilon^\nu + k^2 \epsilon^\nu = 0.$$

This is an identity. Hence, <u>any</u> longitudinal ϵ satisfies Maxwell's eqs., for <u>any</u> k^μ.

Why? The Lagrangian is written in terms of $F_{\mu\nu}$. If $A^\mu(x)$ is <u>pure gauge</u>, ie $A^\mu(x) = \partial^\mu \lambda(x)$ for some $\lambda(x)$, then $F_{\mu\nu} = 0$. Hence, the Lagrangian does not describe pure gauge modes. The longitudinal solutions are pure gauge.

Plane wave decomposition of the vector field

For any K^μ satisfying $K^2=0$ there are two solutions to the transverseness condition $\epsilon \cdot K = 0$. For example, if $K^\mu \propto (1,0,0,1)$ then we can choose

$$\epsilon^{(1)\mu} = \frac{1}{\sqrt{2}}(0,1,i,0)$$
$$\epsilon^{(2)\mu} = \frac{1}{\sqrt{2}}(0,1,-i,0)$$

(It is okay for ϵ^μ to be complex, as long as we make $A^\mu(x)$ real by adding the complex conjugate)

Nonphysical polarizations: $\epsilon^{(3)\mu} = (0,0,0,1)$ — not a sol'n
$\epsilon^{(0)\mu} = (1,0,0,1)$ — Longitudinal

We can normalize the physical solutions such that

$$\boxed{\epsilon^{(r)\mu *} \epsilon^{(s)}_\mu = -\delta^{rs}} \quad , r,s = 1 \text{ or } 2.$$

We can decompose the transverse solutions to Maxwell's equations as follows:

$$A_\mu(x) = \int \frac{d^3 K}{(2\pi)^3 \sqrt{2\omega_K}} \sum_{r=1}^{2} \left(a^r_{\vec{K}} \epsilon^r_\mu(k) e^{-ik\cdot x} + a^{r\dagger}_{\vec{K}} \epsilon^r_\mu(k)^* e^{ik\cdot x} \right)$$

But we need to keep in mind that an arbitrary longitudinal solution $A_\mu(x) \propto \partial_\mu \lambda(x)$ for some $\lambda(x)$ can be added to A_μ and still solve Maxwell's eqs.

Spin of the photon

To figure out the J^z eigenvalue of the particle annihilated by the component of A^μ proportional to $\epsilon^{(2)\mu}$, for example, we can:

Perform a rotation on the field, generated by J^z. If $\epsilon^{(2)} \to e^{-im\theta}\epsilon^{(2)}$ for a rotation by θ about $\hat{x}^3$, then the corresponding particle has spin $J_z = m$.

$$\hat{x}^1 \to \hat{x}^1 \cos\theta + \hat{x}^2 \sin\theta$$
$$\hat{x}^2 \to \hat{x}^2 \cos\theta - \hat{x}^1 \sin\theta$$

$$\vec{\epsilon}^{(2)} = \frac{\hat{x}^1 - i\hat{x}^2}{\sqrt{2}} \to e^{i\theta}\frac{(\hat{x}^1 - i\hat{x}^2)}{\sqrt{2}}$$

Hence, $m = -1$. $\to$ The photon has spin 1.

Similarly, $\vec{\epsilon}^{(1)}$ has $m = +1$.

The solution with $m=0$, $\vec{\epsilon}^{(3)} = \hat{x}^3$, is unphysical. This is due to the massless nature of the photon. ($k^2=0$.) If the photon were massive, i.e. $k^2 = m^2 \neq 0$, then we could go to the rest frame $k = (m, \vec{0})$. Then there are three independent transverse solutions: $\epsilon^{(1)}, \epsilon^{(2)}, \epsilon^{(3)}$. But if $k^2=0$ we have seen that there are only two independent transverse solutions.

Feynman Propagator for Electromagnetic Field

Gauge invariance complicates the canonical quantization procedure. Another approach, functional integral quantization, is better suited to handle gauge theories. We're not going to develop the functional integral just yet.

For now, what we will need is the Feynman propagator for the electromagnetic field. There is an ambiguity because $A^\mu(x)$ and $A^\mu(x) + \partial^\mu \lambda(x)$ describe the same physical field.

We will pick a gauge to eliminate the ambiguity. Given an $A'_\mu(x)$, we can find a $\lambda(x)$ such that $\partial_\mu \partial^\mu \lambda(x) = -\partial^\mu A'_\mu(x)$. Then the gauge transformed field $A^\mu(x) \equiv A'^\mu(x) + \partial^\mu \lambda(x)$ satisfies the Lorenz gauge condition $\boxed{\partial_\mu A^\mu = 0}$.

In Lorenz gauge, Maxwell's equations become $\boxed{\partial_\alpha \partial^\alpha A_\mu = 0}$.

Hence, each component of A^μ satisfies the massless Klein-Gordon equation.

The free scalar field satisfied the Klein-Gordon eqn, and the Feynman propagator satisfied the corresponding Green's function equation, which for the massless field is $\partial_\mu \partial^\mu D_F(x-y) = -i\delta^4(x-y)$.

It is natural to guess that $\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{\pm i g_{\mu\nu}}{k^2 + i\epsilon} e^{-ik\cdot(x-y)} .$$

Consider $\mu = \nu \equiv i$, $i \in 1,2,3$, w/ $x^0 \to y^0$ from the $(x^0 - y^0) > 0$ direction. The propagator is the norm of the state $A_i(y)|0\rangle$, which should be positive.

Exercise: Our expression for the Feynman propagator gives a positive norm iff we choose the negative sign.

We will use this form for the electromagnetic Feynman propagator, which will be okay when inserted in Feynman diagrams, but to really take care of the complications due to gauge invariance we will have to wait a bit for a proper treatment of gauge theories.

Coupling to a Conserved Current:

Suppose J^μ satisfies $\partial_\mu J^\mu = 0$.
Then under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \theta$, $J^\mu \to J^\mu$,

$$A_\mu J^\mu \to (A_\mu + \partial_\mu \theta) J^\mu$$
$$= A_\mu J^\mu + \partial_\mu(\theta J^\mu)$$

Then the action $S = \int d^4x \, A_\mu J^\mu$ is gauge invariant if θJ^μ falls off sufficiently quickly at infinity.

Consider the Lagrangian density $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e A_\mu J^\mu},$
where the coupling constant e
determines the strength of the electromagnetic coupling to J^μ.

Euler-Lagrange Eqs: $\boxed{\partial_\mu F^{\mu\nu} = e J^\nu}$

To connect this to the usual description of the electric and magnetic fields, define

$$F^{0i} = -F^{i0} \equiv -E^i$$
$$-\frac{1}{2} \epsilon^{ijk} F^{jk} = B^i \iff F^{jk} = -\epsilon^{ijk} B^i$$

(summing over spatial indices, $\epsilon^{123} = +1$)

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \boxed{\nabla\cdot\vec{E} = eJ^0 \equiv e\rho}$$

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\frac{\partial E^i}{\partial t} - \epsilon^{jik}\partial_j B^k = eJ^i$$

ie,
$$\boxed{-\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B} = e\vec{J}}$$

We have recovered half of Maxwell's eqs.

The other half are a consequence of the Bianchi ID:

$$\epsilon^{0ijk}\partial_i F_{jk} = +\epsilon^{ijk}\epsilon^{jkl}\partial_i B_l \overset{\text{Exercise}}{\propto} \boxed{\nabla\cdot\vec{B} = 0}$$

$$\epsilon^{i\mu\nu\lambda}\partial_\mu F_{\nu\lambda} = \epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} + \epsilon^{ijk0}\partial_j F_{k0}$$

$$= +\epsilon^{ijk}\epsilon^{jkl}\partial_0 B^l + 2\epsilon^{ijk}\partial_j E^k$$

$$\xrightarrow{\text{Exercise}} \boxed{\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0}$$

Quantum Electrodynamics (QED)

Under the local transformation $\psi \to e^{-ie\theta(x)}\psi$, $\bar{\psi} \to \bar{\psi}e^{ie\theta(x)}$, the Dirac field Lagrangian transforms as

$$\bar{\psi}(i\partial\!\!\!/ - m)\psi \to \bar{\psi}(i\partial\!\!\!/ + e(\partial\!\!\!/\theta) - m)\psi$$

With the addition of a vector field $A^\mu(x)$ such that under the local transformation of $\psi(x)$, $A^\mu(x)$ also transforms $A^\mu(x) \to A^\mu(x) + \partial^\mu\theta(x)$, we can construct an invariant Lagrangian:

$$\bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi \to \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi.$$

We have replaced $\partial_\mu\psi$ in the Dirac Lagrangian with the <u>covariant derivative</u> $\boxed{D_\mu\psi \equiv (\partial_\mu + ieA_\mu)\psi}$

This allowed us to lift the global $U(1)$ symmetry of the theory to a local gauge invariance.

To give dynamics to the vector field we add the <u>gauge kinetic term</u> $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the Lagrangian.

The QED Lagrangian is then,

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi}$$

The electromagnetic coupling to electrons is due to the term $-\bar{\psi}eA\!\!\!/\psi = -eA_\mu\bar{\psi}\gamma^\mu\psi$.