# Properties of the Dirac Field

The Dirac field Hamiltonian is

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k \sum_r \left( a_k^{r\dagger} a_k^r + b_k^{r\dagger} b_k^r \right).$$

The state $a_{k_1}^{r_1\dagger} \cdots a_{k_n}^{r_n\dagger}\, b_{k_{n+1}}^{r_{n+1}\dagger} \cdots b_{k_{n+m}}^{r_{n+m}\dagger} |0\rangle$

has energy $\sum_{i=1}^{n+m} \omega_{k_i}$, as seen by acting on the state w/ $H$.

To determine the other properties of the states created by $a_k^{r\dagger}$ and $b_k^{r\dagger}$ we need to calculate the remaining conserved quantities: the momentum, angular momentum, and global $U(1)$ charge, in particular.

## Spatial Momentum

Consider the spatial translation $\psi(t,\vec{x}) \to \psi(t, \vec{x}+\vec{a})$

$$\frac{\partial \psi}{\partial a_i}\Big|_{a_i=0} = \partial^i \psi$$

$$p^i = \int \Pi_\psi\, \partial^i \psi\, d^3x = -\int \Pi_\psi\, \partial_i \psi\, d^3x = -\int \Pi_\psi (\nabla\psi)_i\, d^3x$$

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger$$

$$\boxed{\vec{P} = -\int d^3x\, i\psi^\dagger \nabla \psi = \int \frac{d^3k}{(2\pi)^3}\, \vec{k} \sum_r \left( a_k^{r\dagger} a_k^r + b_k^{r\dagger} b_k^r \right)}$$

(Exercise)

## Angular Momentum

Consider a rotation by $\theta$ about $\hat{x}^3$ (in Weyl basis):

$$\psi(x) \to \exp\left[-\frac{i}{2}\theta\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\right]\psi(R^{-1}(\theta)\,x)$$

$$R^{-1}(\theta)\,\vec{x} = (x^1+\theta x^2,\ x^2-\theta x^1,\ x^3) + \mathcal{O}(\theta^2)$$

$$\psi(R^{-1}(\theta)\vec{x}) = \psi(x) + \theta\left(x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}\right)\psi(x) + \mathcal{O}(\theta^2)$$

$$\Rightarrow\ \psi(x) \to \psi(x) - \theta\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + \frac{i}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\right)\psi(x) + \mathcal{O}(\theta^2)$$

$$\Pi_\psi \frac{\partial\psi}{\partial\theta}\Big|_{\theta=0} = -i\,\psi^\dagger\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + \frac{i}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\right)\psi$$

Conserved $\hat{z}$-momentum: $J^z = \int \Pi_\psi \frac{\partial\psi}{\partial\theta}\Big|_{\theta=0} d^3x$

$$J^z = \int d^3x\ (-i)\,\psi^\dagger\Big(\underbrace{x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}}_{(\vec{x}\times\nabla)^z} + \frac{i}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\Big)\psi$$

Generalizing to rotation about arbitrary axis,

$$\boxed{\vec{J} = \int d^3x\ \psi^\dagger\left(\vec{x}\times(-i\nabla) + \frac{1}{2}\begin{pmatrix}\vec{\sigma} & \\ & \vec{\sigma}\end{pmatrix}\right)\psi}$$

$$J^z = \int d^3x \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\sum_{rs}\left(a_{k'}^{r\dagger}u^r(k')^\dagger e^{ik'\cdot x} + b_{k'}^r v^r(k')^\dagger e^{-ik'\cdot x}\right)$$

$$\cdot\left((\vec{x}\times(-i\nabla))^z + \frac{1}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}\right)\left(a_k^s u^s(k)e^{-ik\cdot x} + b_k^{s\dagger}v^s(k)e^{ik\cdot x}\right)$$

Consider $J^z a_{\vec{0}}^{r\dagger}|0\rangle$ (rest frame $\vec{k}=\vec{0}$)

The $\vec{x}\times(-i\nabla)$ terms vanish.
Normal ordering as usual, $J^z|0\rangle = 0$, so

$$J^z a_{\vec{0}}^{r\dagger}|0\rangle = [J^z, a_0^{r\dagger}]|0\rangle$$

The only nonvanishing term is from $[a_k^{t\dagger} a_k^s, a_{\vec{0}}^{r\dagger}]$
$= (2\pi)^3 \delta^3(\vec{k})\, a_{\vec{0}}^{t\dagger} \delta^{rs}$

Then $J^z a_{\vec{0}}^{r\dagger}|0\rangle = \frac{1}{2m} \sum_s u^{s\dagger}(\vec{0})\, \frac{1}{2}\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix} u^r(\vec{0})\, a_{\vec{0}}^{s\dagger}|0\rangle$

Use the explicit Weyl basis expressions for $u^{r,s}(\vec{0})$:

$$u^r(\vec{0}) = \sqrt{m}\begin{pmatrix}\xi^r \\ \xi^r\end{pmatrix}, \quad \xi^1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$J^z a_{\vec{0}}^{1\dagger}|0\rangle = \frac{1}{2m}\cdot 2m \sum_s \xi^{s\dagger} \frac{\sigma^3}{2} \xi^1 a_{\vec{0}}^{s\dagger}|0\rangle$$

$$= \xi^{1\dagger} \frac{\sigma^3}{2} \xi^1 a_{\vec{0}}^{1\dagger}|0\rangle$$

$$= \frac{1}{2} a_{\vec{0}}^{1\dagger}|0\rangle$$

$$J^z a_{\vec{0}}^{2\dagger}|0\rangle = \sum_s \xi^{s\dagger} \frac{\sigma^3}{2} \xi^2 a_{\vec{0}}^{s\dagger}|0\rangle$$

$$= \xi^{2\dagger} \frac{\sigma^3}{2} \xi^2 a_{\vec{0}}^{s\dagger}|0\rangle$$

$$= -\frac{1}{2} a_{\vec{0}}^{2\dagger}|0\rangle$$

Similarly, $J^z b_{\vec{0}}^{1\dagger}|0\rangle = -\frac{1}{2}\, b_{\vec{0}}^{1\dagger}|0\rangle$

$$J^z b_{\vec{0}}^{2\dagger}|0\rangle = +\frac{1}{2}\, b_{\vec{0}}^{2\dagger}|0\rangle$$

Hence, $a_{\vec{0}}^{r\dagger}$ creates particles w/ angular momentum $\pm 1/2$ in the rest frame. This is identified with the spin angular momentum of the Dirac particle.

$b_{\vec{0}}^{r\dagger}$ creates particles w/ the opposite spin.

U(1) Charge

The conserved current due to the symmetry $\psi \to e^{i\theta}\psi$, $\bar{\psi} \to e^{-i\theta}\bar{\psi}$ is $J^\mu = -\bar{\psi}\gamma^\mu\psi$.

The conserved charge is:

$$Q = \int d^3x\, J^0 = -\int d^3x\, \psi^\dagger \psi$$

$$= -\int \frac{d^3k}{(2\pi)^3} \sum_r \left( a_k^{r\dagger} a_k^r - b_k^{r\dagger} b_k^r \right) + \text{infinite const.}$$

(Exercise.)

Normal ordering eliminates the infinite const. Then, replacing $Q$ with $:Q:$,

$$Q\, a_k^{r\dagger}|0\rangle = -a_k^{r\dagger}|0\rangle$$

$$Q\, b_k^{r\dagger}|0\rangle = +b_k^{r\dagger}|0\rangle$$

To summarize: $a_{\vec{k}}^{r\dagger}$ creates particles w/ energy $\omega_k$, momentum $\vec{k}$, spin $\pm 1/2$, charge $-1$. $b_{\vec{k}}^{r\dagger}$ creates particles w/ same energy, momentum, opposite spin, charge.

The Dirac Propagator: Will be important in the calculation of scattering amplitudes.

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|\int\frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\sum_{rs}\left(a_k^r u^r(k)e^{-ik\cdot x}+b_k^{r\dagger}v^r(k)e^{ik\cdot x}\right)$$

$$\cdot\left(a_{k'}^{s\dagger}\bar{u}^s(k')e^{ik'\cdot y}+b_{k'}^s\bar{v}^s(k')e^{-ik'\cdot y}\right)|0\rangle$$

$$=\langle 0|\int\frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\sum_{rs}a_k^r u^r(k)a_{k'}^{s\dagger}\bar{u}^s(k')e^{-i(k\cdot x-k'\cdot y)}|0\rangle$$

All other terms vanish because $b_{k'}^s|0\rangle$ and $\langle 0|b_k^{r\dagger}=0$.

$$=\langle 0|\int\frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\sum_{rs}\{a_k^r,a_{k'}^{s\dagger}\}u^r(k)\bar{u}^s(k')e^{-i(k\cdot x-k'\cdot y)}|0\rangle$$

$$=\langle 0|\int\frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\sum_{rs}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta_{rs}\,u^r(k)\bar{u}^s(k')e^{-i(k\cdot x-k'\cdot y)}|0\rangle$$

$$=\langle 0|\int\frac{d^3k}{(2\pi)^3 2\omega_k}\sum_r u^r(k)\bar{u}^r(k)e^{-ik\cdot(x-y)}|0\rangle$$

$$=\langle 0|\int\frac{d^3k}{(2\pi)^3 2\omega_k}(\not{k}+m)e^{-ik\cdot(x-y)}|0\rangle$$

$$=(i\not{\partial}_x+m)\langle 0|\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot(x-y)}|0\rangle$$

$$=(i\not{\partial}_x+m)\underbrace{\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot(x-y)}}$$

This was $\langle 0|\phi(x)\phi(y)|0\rangle$ for the real scalar field

$\equiv D(x-y)$

Similarly, $\langle 0|\bar{\Psi}_b(y)\,\Psi_a(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_r v_a^r(k)\,\bar{v}_b^r(k) \cdot e^{ik\cdot(x-y)}$

a, b label components of $\Psi$,
a, b = 1, …, 4

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (\not{k} - m)_{ab}\, e^{ik\cdot(x-y)}$$

$$= -(i\not{\partial}_x + m)_{ab} \underbrace{\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik\cdot(x-y)}}_{\text{This was } \langle 0|\phi(y)\phi(x)|0\rangle \text{ for the real scalar field}}$$

Feynman Propagator

We define the time ordering operation for fermions with a $-$ve sign when fermion fields are exchanged:

$$\langle 0|T\,\Psi(x)\,\bar{\Psi}(y)|0\rangle = \begin{cases} \langle 0|\Psi(x)\,\bar{\Psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\Psi}(y)\,\Psi(x)|0\rangle & x^0 < y^0 \end{cases}$$

With this extra $-$ve sign, $\boxed{\begin{aligned}&\langle 0|T\,\Psi(x)\,\bar{\Psi}(y)|0\rangle \\ &= (i\not{\partial}_x + m)\,\langle 0|T\,\phi(x)\,\phi(y)|0\rangle\end{aligned}}$

where $\langle 0|T\,\phi(x)\,\phi(y)|0\rangle$ is the Feynman propagator for a real scalar field.

Recall that we can also write $\langle 0|T\,\phi(x)\,\phi(y)|0\rangle$ as,

$$\langle 0|T\,\phi(x)\,\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\,e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \equiv D_F(x-y)$$

So, $$\boxed{\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} e^{-ik\cdot(x-y)}}$$

Note that $(i\not{\partial}_x - m)\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle$

$$= \int \frac{d^4k}{(2\pi)^4} \, i\frac{(k^2-m^2)\mathbb{1}}{k^2-m^2+i\epsilon} e^{-ik\cdot(x-y)}$$

$$= i\,\delta^4(x-y)\,\mathbb{1}$$

Hence, $-i\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle$ is a Green's function for the Dirac equation.