Dirac Field Bilinears

If $\Psi(x)$ is a Dirac field and Γ is a 4×4 matrix, then $\bar{\Psi}\Gamma\Psi$ can be decomposed into a sum of terms which transform covariantly under Lorentz transformations.

$\leftarrow$ antisymmetrized in $\mu \leftrightarrow \nu$

The matrices $\mathbb{1}$, γ^μ, $\gamma^{[\mu}\gamma^{\nu]} \equiv \frac{1}{2}[\gamma^\mu,\gamma^\nu] = -i\sigma^{\mu\nu}$, $\gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$, $\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$ form a complete basis of 4×4 matrices. The Lorentz transformation of $\bar{\Psi}\Gamma\Psi$ can be read directly from the index structure of Γ. $\gamma^{\mu\nu\rho}$ and $\gamma^{\mu\nu\rho\sigma}$ can be simplified by defining $\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3}$.

<u>Exercise</u>: Show $\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$

and $\gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$

Note that under Lorentz transfs. connected to the identity, $\bar{\Psi}\gamma^{\mu\nu\rho\sigma}\Psi$ is a scalar.

Properties of γ^5:

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$$

Recall $\gamma^0 = \beta = (\gamma^0)^\dagger$

$$\gamma^i = \beta\alpha^i \longrightarrow (\gamma^i)^\dagger = (\alpha^i)^\dagger\beta^\dagger = \alpha^i\beta = -\beta\alpha^i = -\gamma^i$$

So, $(\gamma^5)^\dagger = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$

$$\Longrightarrow \boxed{(\gamma^5)^\dagger = \gamma^5}$$

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\ \gamma^0\gamma^1\gamma^2\gamma^3$$
$$= -\gamma^0\gamma^0\ \gamma^1\gamma^1\ \gamma^2\gamma^2\ \gamma^3\gamma^3$$
$$= +\mathbb{1}$$

$$\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5$$
$$= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= 0$$

(γ^μ is anticommuted through three of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$.)

$$[\gamma^5, \sigma^{\mu\nu}] = \frac{i}{2}\left(\gamma^5[\gamma^\mu, \gamma^\nu] - [\gamma^\mu, \gamma^\nu]\gamma^5\right)$$
$$= 0$$

(anticommuting γ^5 through γ^μ and γ^ν)

γ^5 and Chirality

Because $\frac{i}{4}\sigma^{\mu\nu}$ generates Lorentz transformations and $[\gamma^5, \sigma^{\mu\nu}] = 0$, Dirac spinors Ψ with different γ^5 eigenvalues transform without mixing.

The 4-component Dirac spinor representation decomposes into 2-component reps with $\gamma^5 = \pm 1$.

In the Weyl basis, $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

$$\sigma^{ij} = \frac{i}{2}\begin{pmatrix} -[\sigma^i, \sigma^j] & \\ & -[\sigma^i, \sigma^j] \end{pmatrix} = \sum_k \epsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

(convention: $\epsilon^{123} = 1$)

Infinitessimal rotation parametrized by

$$\omega^{ij} = \sum_k \epsilon^{ijk}\theta^k$$

$$-\frac{i}{4}\omega^{ij}\sigma_{ij} = -\frac{i}{4}\sum_{ijkl}\epsilon^{ijk}\theta^k\epsilon^{ijl}\begin{pmatrix}\sigma^l & \\ & \sigma^l\end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix}\vec{\theta}\cdot\vec{\sigma} & \\ & \vec{\theta}\cdot\vec{\sigma}\end{pmatrix}$$

using $\sum_{ij}\epsilon^{ijl}\epsilon^{ijk} = 2\delta^{kl}$

$$\sigma_{0i} = \frac{i}{2}[\gamma_0, \gamma_i] = -\frac{i}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\right]$$

$$= i\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}$$

Infinitessimal boost parametrized by $\omega^{0i} = -\frac{v^i}{c}$

$$-\frac{i}{4}\left(\omega^{0i}\sigma_{0i} + \omega^{i0}\sigma_{i0}\right) = -\frac{i}{2}\omega^{0i}\sigma_{0i}$$

$$= -\frac{i}{2}\left(-\frac{v^i}{c}\right)\begin{pmatrix}i\sigma^i & 0\\ 0 & -i\sigma^i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-\frac{\vec{v}}{c}\cdot\vec{\sigma} & 0\\ 0 & \frac{\vec{v}}{c}\cdot\vec{\sigma}\end{pmatrix}$$

$$\exp\left[-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right] = \exp\left[\frac{\vec{\sigma}}{2}\cdot\begin{pmatrix}-i\vec{\theta}-\vec{v}/c & 0\\ 0 & -i\vec{\theta}+\vec{v}/c\end{pmatrix}\right]$$

In the Weyl basis, write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

where ψ_L, ψ_R each have two components.

Under a Lorentz transformation,

$$\psi \to \exp\left[-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right]\psi \quad , \quad \text{i.e.}$$

$$\psi_L \to \exp\left[-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta} - \frac{\vec{v}\cdot\vec{\sigma}}{2c}\right]\psi_L$$

$$\psi_R \to \exp\left[-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta} + \frac{\vec{v}\cdot\vec{\sigma}}{2c}\right]\psi_R$$

ψ_L and ψ_R transform in different irreducible representations of the Lorentz group.

In the Weyl basis, $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

$\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ has γ^5 eigenvalue -1. — called left-handed.

$\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$ has γ^5 eigenvalue $+1$. — called right-handed.

Free Particle Solutions to the Dirac Equation

We will construct plane-wave solutions in the Weyl basis.

$$\psi(x) = u(p)\, e^{-ip\cdot x}, \quad p^\mu p_\mu = m^2$$

Positive frequency solutions: $p^0 > 0$.

Dirac equation: $(\gamma^\mu p_\mu - m)\, u(p) = 0$

In the rest frame of the particle, $p^0 = m$, $p^i = 0$, $i = 1, 2, 3$. Then the Dirac equation becomes $m(\gamma^0 - 1)u(p) = 0$.

$$m\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p) = 0$$

Solutions: $u(p) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$

ξ = arbitrary 2-component constant spinor.
Convenient normalization: $\xi^\dagger \xi = 1$.

To find an arbitrary free particle solution, boost the rest frame solution in the direction of motion.

Example: Boost along x^3-direction.

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left[\omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix} = (\cosh\omega\, \mathbb{1} + \sinh\omega\, \sigma^1)\begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m\cosh\omega \\ m\sinh\omega \end{pmatrix}$$

Boost ψ: $\psi(x) = \exp\left[-\frac{\omega}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\, e^{-ip\cdot x}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$

$$= \left[\cosh\frac{\omega}{2}\,\mathbb{1} - \sinh\frac{\omega}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right]\sqrt{m}\, e^{-ip\cdot x}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \sqrt{m}\, e^{-ip\cdot x}\begin{pmatrix}\left(\cosh\frac{\omega}{2}\mathbb{1} - \sinh\frac{\omega}{2}\sigma^3\right)\xi\\ \left(\cosh\frac{\omega}{2}\mathbb{1} + \sinh\frac{\omega}{2}\sigma^3\right)\xi\end{pmatrix}$$

Use $\cosh\frac{\omega}{2} = \sqrt{\frac{1+\cosh\omega}{2}}$, $\sinh\frac{\omega}{2} = \sqrt{\frac{\cosh\omega - 1}{2}}$,

$\cosh\omega = E/m$

$\Rightarrow \sqrt{m}\cosh\frac{\omega}{2} = \sqrt{\frac{m+E}{2}}$, $\sqrt{m}\sinh\frac{\omega}{2} = \sqrt{\frac{E-m}{2}}$

$$\psi(x) = \frac{e^{-ip\cdot x}}{\sqrt{2}}\begin{pmatrix}\left(\sqrt{E+m}\,\mathbb{1} - \sqrt{E-m}\,\sigma^3\right)\xi\\ \left(\sqrt{E+m}\,\mathbb{1} + \sqrt{E-m}\,\sigma^3\right)\xi\end{pmatrix}$$

<u>Example</u>: If $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ (spin-up along $\hat{x}^3$)

$$\psi_+(x) = \frac{e^{-ip\cdot x}}{\sqrt{2}}\begin{pmatrix}\sqrt{E+m} - \sqrt{E-m}\\ 0\\ \sqrt{E+m} + \sqrt{E-m}\\ 0\end{pmatrix}$$

Use $\left(\sqrt{E+m} - \sqrt{E-m}\right)^2 = E+m+E-m - 2\sqrt{E^2-m^2}$
$$= 2(E - p^3)$$
and $\left(\sqrt{E+m} + \sqrt{E-m}\right)^2 = E+m+E-m + 2\sqrt{E^2-m^2}$
$$= 2(E+p^3)$$

$$\rightarrow \boxed{\psi_+(x) = \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix} e^{-ip\cdot x}}$$

Similarly, with $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\boxed{\psi_-(x) = \begin{pmatrix} 0 \\ \sqrt{E+p^3} \\ 0 \\ \sqrt{E-p^3} \end{pmatrix} e^{-ip\cdot x}}$$

ψ_+ and ψ_- are eigenstates of the <u>helicity operator</u>:

$$h \equiv \hat{p} \cdot \frac{\vec{S}}{\hbar} = \frac{1}{2} \hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$\boxed{h\psi_+ = \frac{1}{2}\psi_+} \quad , \quad \boxed{h\psi_- = -\frac{1}{2}\psi_-}$$

In the massless limit or large energy, $E \approx p^3$

$$\psi_-(x) \approx e^{-ip\cdot x} \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_+(x) \approx e^{-ip\cdot x} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix}$$

(Now we see the reason for normalizing the solutions w/ the factor of $\sqrt{m}$ — so the massless limit is well behaved.)

In the massless limit the <u>helicity</u> eigenstates become <u>chirality</u> eigenstates, w/ ψ_- left-handed and ψ_+ right-handed.

The positive frequency solutions to the Dirac equation can be written in a form valid for arbitrary 3-momentum:

$\Psi(x) = u(p) e^{-ip\cdot x}$, $p^\mu p_\mu = m^2$, $p^0 > 0$, with two linearly independent solutions for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p^\mu \sigma_\mu}\, \xi^s \\ \sqrt{p^\mu \bar{\sigma}_\mu}\, \xi^s \end{pmatrix}} \quad , \; s = 1, 2$$

$\sigma^\mu = (1, \vec{\sigma})^\mu$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})^\mu$,

$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Normalization: $u^{r\dagger}(p) u^s(p) = \xi^{r\dagger} p^\mu \sigma_\mu \xi^s + \xi^{r\dagger} p^\mu \bar{\sigma}_\mu \xi^s$

$$= 2p^0 \delta^{rs} \equiv 2\omega_{\vec{p}}\, \delta^{rs}$$

where $\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$

Equivalently, $\bar{u}^r(p) u^s(p) = \xi^{r\dagger} (p^\mu \sigma_\mu)^{1/2} (p^\nu \bar{\sigma}_\nu)^{1/2} \xi^s$

$$+ \xi^{r\dagger} (p^\mu \bar{\sigma}_\mu)^{1/2} (p^\nu \sigma_\nu)^{1/2} \xi^s$$

$$= 2\, \xi^{r\dagger} \Big((p^0)^2 - \sum_{ij} p^i p^j \sigma^i \sigma^j \Big)^{1/2} \xi^s$$

$$= 2\, \xi^{r\dagger} \big((p^0)^2 - \vec{p}^{\,2} \big)^{1/2} \xi^s$$

$$= 2m\, \delta^{rs}$$

$$\longrightarrow \boxed{\bar{u}^r(p) u^s(p) = 2m\, \delta^{rs}}$$

Negative-Frequency Solutions

$\Psi(x) = v(p) e^{ip\cdot x}$, $p^\mu p_\mu = m^2$, $p^0 > 0$ (but note positive sign in exponent)

Rest frame: $p^0 = m$, $\vec{p} = 0$.
Dirac equation: $-(\gamma^\mu p_\mu + m) v(p) = 0$

$$-m \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(p) = 0$$

Solutions: $v(p) = \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}$, η = constant 2-component spinor

In the Weyl basis the Lorentz transformations act independently on the top two and bottom two components, so we can read off the boosted solutions from the positive-frequency solutions.

$$\rightarrow \boxed{v^s(p) = \begin{pmatrix} \sqrt{p^\mu \sigma_\mu}\ \eta^s \\ -\sqrt{p^\mu \bar{\sigma}_\mu}\ \eta^s \end{pmatrix}}, \quad s = 1,2, \quad \eta^s = \text{basis of 2-component spinors.}$$

Normalization: $v^{r\dagger}(p) v^s(p) = 2\omega_{\vec{p}}\, \delta^{rs}$

Equivalently $\boxed{\bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs}}$

Furthermore, the positive and negative-frequency solutions are orthogonal in the sense that

$$\boxed{\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0}$$