PHYS 63D S'24 Problem Set 9 Solutions

6.2 a) Escape velocity:

$$\underbrace{\frac{m v_E^2}{2}}_{\text{kinetic energy}} - \underbrace{\frac{GMm}{R}}_{\text{gravitational energy relative to } r=\infty} = 0$$

$$v_E = \sqrt{\frac{2GM}{R}} = c$$

$$\rightarrow \boxed{R = \frac{2GM}{c^2}}$$

b) $\Delta S = S_2 - 2S_1 = \frac{k_B c^3}{4G\hbar}(A_2 - 2A_1)$

$$= \frac{k_B c^3}{4G\hbar} \cdot \left(4\pi R_2^2 - 2 \cdot 4\pi R_1^2\right)$$

$$= \frac{k_B c^3}{4G\hbar} \cdot 4\pi \left(\left(\frac{2G \cdot 2M}{c^2}\right)^2 - 2\left(\frac{2GM}{c^2}\right)^2\right)$$

$$\boxed{\Delta S = \frac{8\pi G k_B M^2}{c\hbar} > 0}$$

$M = M_\odot = 2 \times 10^{30}$ kg

$$\rightarrow \Delta S = \frac{8\pi \cdot (6.7 \times 10^{-11}\ \mathrm{Nm^2/kg^2})(1.4 \times 10^{-23}\ \mathrm{J/K})(2 \times 10^{30}\ \mathrm{kg})^2}{(3 \times 10^8\ \mathrm{m/s})(1.05 \times 10^{-34}\ \mathrm{J \cdot s})}$$

$$\boxed{\Delta S \approx 3 \times 10^{54}\ \mathrm{J/K}}$$

Entropy in N_I bits is $S = k_B \ln(2^N)$

↑ # configurations

$$\rightarrow N_I = \frac{S}{k_B \ln 2}$$

Information lost: $N_I = \frac{\Delta S}{k_B \ln 2} = \frac{(3\times10^{54}\ J/K)}{(1.4\times10^{-23}\ J/K)\cdot \ln 2}$

$$\boxed{N_I \approx 3\times10^{77}}$$

c) $E = Mc^2$

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{c^2}\frac{\partial}{\partial M}\left(\frac{k_B c^3}{4G\hbar}\cdot 4\pi\left(\frac{2GM}{c^2}\right)^2\right)$$

$$= \frac{8\pi k_B G M}{\hbar c^3}$$

$$\rightarrow \boxed{T = \frac{\hbar c^3}{8\pi k_B G M}} = T_H$$

↑ temperature of Hawking radiation.

d) Stefan-Boltzmann Law:

$$\frac{\partial E}{\partial t} = -A\sigma T^4, \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

$$A = 4\pi\left(\frac{2GM}{c^2}\right)^2, \quad T = \frac{\hbar c^3}{8\pi k_B G M}$$

$$\rightarrow \boxed{\frac{\partial E}{\partial t} = -\frac{\hbar c^6}{(15{,}360)G^2\pi M^2}}$$

e) $E = Mc^2 \rightarrow \frac{dM}{dt} = -\frac{\hbar c^4}{15{,}360\, G^2\pi M^2} \equiv -a/M^2$, where $a = \frac{\hbar c^4}{15{,}360\, G^2 \pi}$

$$\rightarrow \left(\frac{M^3}{3} - \frac{M_0^3}{3}\right) = -at$$

$$\rightarrow M(t) = \left(M_0^3 - 3at\right)^{1/3}$$

$M(\tau)=0$ when $\tau = \frac{M_0^3}{3a} = \frac{5120 G^2 \pi M_\odot^3}{\hbar c^4} \approx 10^{75}$ s

(compare with the current age of the universe $\sim 10^{18}$ s.

f) $M = \frac{\hbar c^3}{8\pi k_B G T}$. With $T = 2.7$ K,

$$\boxed{M \approx 4.5 \times 10^{22} \text{ kg}}$$

g) Crude argument: If a region of space could have $S > S_{BH}$ for that region then adding matter to form a black hole would decrease the overall entropy, in violation of the 2nd Law.

Hence, $S \leq S_{BH} = \frac{k_B c^3}{4 G \hbar} A$

$A = 4\pi R^2$ = area enclosing the region.