PHYS 630 S'24 Problem Set 7 Solutions

Kardar

4.8 $\vec{B} = B\hat{z}$, $M_z = \mu \sum_{i=1}^{N} m_i$, $m_i \in \{-s, -s+1, \dots, s-1, s\}$

a) $Z = \sum_{\{m_i\}} \exp(\beta \vec{B} \cdot \vec{M})$

$$= \sum_{\{m_i\}} \exp\left(\beta B \mu \sum_{i=1}^{N} m_i\right) = \left[\sum_{m_i=-s}^{s} \exp(\beta \mu B m_i)\right]^N$$

$$= \left[\left(\exp(\beta\mu B)\right)^{-s} + \left(\exp(\beta\mu B)\right)^{-s+1} + \dots + \left(\exp(\beta\mu B)\right)^{s}\right]^N$$

$$= \left[\frac{\left(\exp(\beta\mu B)\right)^{-s} - \left(\exp(\beta\mu B)\right)^{s+1}}{1 - \exp(\beta\mu B)}\right]^N$$

$$= \left[\frac{\exp\left(-\beta B\mu(s+1/2)\right) - \exp\left(\beta B\mu(s+1/2)\right)}{\exp(-\beta B\mu/2) - \exp(\beta B\mu/2)}\right]^N$$

$$\boxed{Z = \left(\frac{\sinh\left(\beta B\mu(s+1/2)\right)}{\sinh(\beta B\mu/2)}\right)^N}$$

b) $G = E - BM = -k_B T \ln Z$

$$= -Nk_B T \ln\left[\sinh\left(\beta B\mu(s+1/2)\right)\right] + Nk_B T \ln\left[\sinh\left(\beta B\mu/2\right)\right]$$

Use $\sinh x \approx x + \frac{x^3}{3!}$

and $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x \ll 1$

$$\Rightarrow G \approx -Nk_B T \ln(2s+1) - Nk_B T (\beta B\mu)^2 \frac{(s^2+s)}{6}$$

for $\beta B\mu \ll 1$

$$\boxed{G \approx \text{const.} - \frac{NB^2\mu^2 s(s+1)}{6k_B T} + O(B^4)}$$

c) $\chi = \left.\frac{\partial M_z}{\partial B}\right|_{B=0}$

$$\langle M_z \rangle = k_B T \frac{\partial \ln Z}{\partial B} = -\frac{\partial G}{\partial B} \approx \frac{NB\mu^2 s(s+1)}{3k_B T}$$

$$\boxed{\chi} = \left.\frac{\partial \langle M_z \rangle}{\partial B}\right|_{B=0} \boxed{= \frac{N\mu^2 s(s+1)}{3k_B T}}$$

$$= \frac{C}{T}, \text{ where } \boxed{C = \frac{N\mu^2 s(s+1)}{3k_B}}$$

10. $\mathcal{H} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} - \mu B S_i^z \right]$

a) $Z = \sum_{m_s} \exp[-\beta \mathcal{H}] = \frac{1}{N!} \left(\frac{V}{\lambda^3} \left(e^{\beta \mu B} + 1 + e^{-\beta \mu B} \right) \right)^N$

$= \frac{1}{N!} \left(\frac{V}{\lambda^3} (2 \cosh(\beta \mu B) + 1) \right)^N$

where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$

b) $p(S_i^z = -1) = \frac{e^{-\beta \mu B}}{2 \cosh(\beta \mu B) + 1}$

$p(S_i^z = 0) = \frac{1}{2 \cosh(\beta \mu B) + 1}$

$p(S_i^z = +1) = \frac{e^{\beta \mu B}}{2 \cosh(\beta \mu B) + 1}$

c) $M = \mu \sum_{i=1}^{N} S_i^z$

$\langle M \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial B} = \frac{1}{\beta} \cdot N \frac{\partial}{\partial B} \ln(2 \cosh(\beta \mu B) + 1)$

$$\boxed{\langle M \rangle = N\mu \left(\frac{2 \sinh(\beta \mu B)}{2 \cosh(\beta \mu B) + 1} \right)}$$

d) $$\chi = \frac{\partial \langle M \rangle}{\partial B}\bigg|_{B=0}$$

$$= N\beta\mu^2 \left(\frac{2\cosh(\beta\mu B)}{2\cosh(\beta\mu B)+1} - \frac{4\sinh(\beta\mu B)\sinh(\beta\mu B)}{(2\cosh(\beta\mu B)+1)^2} \right)\bigg|_{B=0}$$

$$\boxed{\chi = N\beta\mu^2 \cdot \frac{2}{3}}$$