PHYS 630 S'24 Problem Set 6 Solutions

Kardar

4.5 $\mathcal{H} = \sum_{i=1}^{N} A\,|\vec{p}_i|^s$, d dimensions

a) $Z(N,T,V) = \frac{1}{N!\,h^{dN}} \int d^d q_1 \cdots d^d q_N\, d^d p_1 \cdots d^d p_N \exp[-\beta \mathcal{H}]$

$= \frac{1}{N!} \left[\int \frac{d^d q_1\, d^d p_1}{h^d} \exp\left[-\beta A |\vec{p}_1|^s\right] \right]^N$

$= \frac{1}{N!} \frac{V^N}{h^{dN}} \left[\int d^d p_1 \exp\left(-\beta A |\vec{p}_1|^s\right) \right]^N$

$= \frac{1}{N!} \frac{V^N}{h^{dN}} \left[\int d\Omega_{d-1} \int_0^\infty dp_1\, p_1^{d-1} \exp\left(-\beta A p_1^s\right) \right]^N$

$\uparrow$ S_d (Kardar's notation)

$= \frac{1}{N!} \frac{V^N S_d^N}{h^{dN}} \frac{1}{(\beta A)^{dN/s}} \left[\int_0^\infty dx\, x^{d-1} e^{-x^s} \right]^N$

$\swarrow$

$= \frac{1}{N!} \frac{V^N S_d^N}{h^{dN}} (\beta A)^{-dN/s} \cdot \left(\frac{1}{s} \Gamma(d/s) \right)^N$

b) $F = E - TS = -k_B T \ln Z$

$dF = -S dT - P dV$

$$P = -\left.\frac{\partial F}{\partial V}\right|_T = k_B T \frac{\partial \ln Z}{\partial V} = \frac{N k_B T}{V}$$

$$\rightarrow \boxed{PV = N k_B T}$$

$$\boxed{E = -\left.\frac{\partial \ln Z}{\partial \beta}\right|_V = \frac{d}{s}\frac{N}{\beta} = \frac{d}{s} N k_B T}$$

Additional Problem

N diatomic molecules, each along $\hat{x}$ or $\hat{z}$ w/ energy 0, or along $\hat{y}$ w/ energy ε.

a) $Z = \frac{1}{N!}\left(2 + e^{-\beta\varepsilon}\right)^N$

b) $F = E - TS = -k_B T \ln Z$

$\approx k_B T\left(N \ln \frac{N}{e}\right) - N k_B T \ln\left(2 + e^{-\beta\varepsilon}\right)$

c) $E = -\frac{\partial \ln Z}{\partial \beta} = \frac{N\varepsilon e^{-\beta\varepsilon}}{2 + e^{-\beta\varepsilon}}$

d) $dF = -S\,dT - P\,dV$

$\rightarrow S = -\left(\frac{\partial F}{\partial T}\right)_V$

$= -k_B\left(N \ln N/e\right) + N k_B\left(\ln\left(2 + e^{-\beta\varepsilon}\right) + \frac{\left(\frac{\varepsilon}{k_B T}\right) e^{-\beta\varepsilon}}{2 + e^{-\beta\varepsilon}}\right)$

e) $C = \frac{dE}{dT} = N k_B \left(\frac{\varepsilon}{k_B T}\right)^2 \frac{e^{-\beta\varepsilon}}{\left(2 + e^{-\beta\varepsilon}\right)^2}$

f) prob(molecule along $\hat{z}$)

$= \frac{e^{-\beta\varepsilon}}{2 + e^{-\beta\varepsilon}}$