PHYS 630 S'24 Problem Set 3 Solutions

Kardar, Chapter 1

4. a) $dE = TdS - PdV$. We want to consider $E(T,V)$, but from the form of dE, $E(S,V)$ is more natural.

Use the chain rule for differentials:

$$dE = T\left(\frac{\partial S}{\partial T}\bigg|_V dT + \frac{\partial S}{\partial V}\bigg|_T dV\right) - PdV$$

$$= T\frac{\partial S}{\partial T}\bigg|_V dT + \left(T\frac{\partial S}{\partial V}\bigg|_T - P\right) dV$$

$$\Rightarrow \frac{\partial E}{\partial V}\bigg|_T = T\frac{\partial S}{\partial V}\bigg|_T - P$$

We can relate $\frac{\partial S}{\partial V}\big|_T$ to P, V, and T by a Maxwell relation:

$F = E - TS$, $dF = -SdT - PdV$

$$\rightarrow -\frac{\partial S}{\partial V}\bigg|_T = -\frac{\partial P}{\partial T}\bigg|_V$$

Then, $$\frac{\partial E}{\partial V}\bigg|_T = T\frac{\partial P}{\partial T}\bigg|_V - P$$

With $PV = Nk_BT$, $\boxed{\frac{\partial E}{\partial V}\bigg|_T = T\cdot\frac{Nk_B}{V} - P = 0}$

Hence $\boxed{E(T,V) = E(T)}$

b) Reversing the logic of part (a),

$$\frac{\partial E}{\partial V}\bigg|_T = 0 \rightarrow T\frac{\partial P}{\partial T}\bigg|_V - P = 0$$

General Solution: $\boxed{P(T,V) = f(V)T}$ for some function $f(V)$.

Kardar Chapter 2

4. a) $p(r)$ = prob. density for real random variable $r \in [0,1]$

$p_n(x)$ = prob. density for the largest value of a set of n random #s $\{r_\alpha\}$, $\alpha = 1, \dots, n$ each drawn from $p(r)$.

$$p_n(x)\,dx = n\, p(r_1 = x,\ r_2 < x,\ r_3 < x, \dots, r_n < x)\, dx$$

n choices for which r_α is the maximum

$$= n\, p(r_1 = x) \prod_{\alpha=2}^{n} p(r_\alpha < x)\ dx$$

$$= \boxed{n\, p(x) \left[\int_0^x p(r)\, dr \right]^{n-1}}\ dx$$

b) Uniform distribution with $r \in [0,1]$: $p(r) = \begin{cases} 1, & r \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

For $x \in [0,1]$:

$$p_n(x)\,dx = n \cdot 1 \cdot x^{n-1}\ dx$$

$$\boxed{p_n(x) = n\, x^{n-1}}$$

$$\langle x \rangle_c = \langle x \rangle = \int_0^1 dx\ x\, p_n(x)\,dx = \int_0^1 n x^n\, dx = \frac{n x^{n+1}}{n+1}\Big|_0^1$$

$$\boxed{\langle x \rangle_c = \frac{n}{n+1}} \qquad \text{mean}$$

$$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 \qquad \text{variance}$$

$$= \int_0^1 x^2 p_n(x)\, dx - \left(\frac{n}{n+1}\right)^2 = \int_0^1 n x^{n+1}\, dx - \left(\frac{n}{n+1}\right)^2$$

$$= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \boxed{\frac{n}{(n+2)(n+1)^2}}$$

9. a) Deposition rate = d layers/sec.

Poisson Process with rate d over time t:

$P_{dt}(m)$ = prob(m atoms deposited over time t)

$$= \boxed{\frac{(dt)^m e^{-dt}}{m!}}$$

Fraction of glass not covered = prob(0 atoms deposited)

$$\boxed{P_{dt}(0) = e^{-dt}}$$

b) For a Poisson process with PDF given by $P_{dt}(m)$, all cumulants are equal, $\langle m^n \rangle_c = dt$.

Hence, the variance $\boxed{\langle m^2 \rangle_c = dt}$

Check: $\langle m \rangle = \sum_{m=0}^{\infty} m \frac{(dt)^m}{m!} e^{-dt} = dt \left(\sum_{m=0}^{\infty} \frac{(dt)^m}{m!} \right) e^{-dt}$

$$= dt\, e^{dt} e^{-dt} = dt$$

$$\langle m^2 \rangle = \sum_{m=0}^{\infty} m^2 \frac{(dt)^m}{m!} e^{-dt}$$

$$= (dt)^2 \sum_{m=0}^{\infty} \frac{(dt)^m}{m!} e^{-dt} + dt \sum_{m=0}^{\infty} \frac{(dt)^m}{m!} e^{-dt} = (dt)^2 + dt$$