

Degenerate Bose Gas

Avg. Bose occupation number:

$$\langle n_{\vec{k}} \rangle_+ = \frac{1}{e^{\beta P} [\beta(\epsilon(\vec{k}) - \mu)] - 1}$$

$$\langle n_{\vec{k}} \rangle_+ \geq 0 \rightarrow \mu < \min(\epsilon(\vec{k})) = 0$$

High-T, low density: $\mu \rightarrow -\infty$, $z \rightarrow 0$

low-T, high density: $\mu \rightarrow 0$, $z \rightarrow 1$

From the general discussion of the nonrelativistic quantum gas:

$$\left\{ \begin{array}{l} P_+ = \frac{g}{\lambda^3} f_{3/2}^+(z) \\ n_+ = \frac{g}{\lambda^3} f_{1/2}^+(z) \\ E_+ = \frac{7}{2} P_+ \end{array} \right.$$

where $f_m^+(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1}$

As $z \rightarrow 0$,

$$\frac{n_+ \lambda^3}{g} = f_{1/2}^+(z) \rightarrow z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots$$

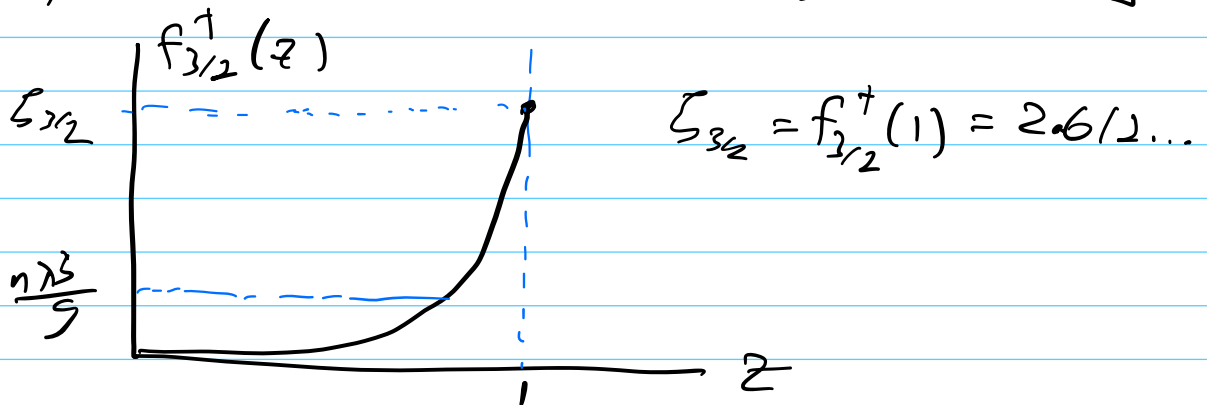
$$\rightarrow z = \frac{n_+ \lambda^3}{g} - \frac{1}{2^{3/2}} \left(\frac{n_+ \lambda^3}{g} \right)^2 - \dots$$

High- T , $z \ll 1$:

$$\frac{\beta P + \lambda^3}{g} = f_{5/2}^+(z) = z + \frac{z^2}{2^{5/2}} + \dots$$
$$= \frac{n + \lambda^3}{g} + \left(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}} \right) \left(\frac{n + \lambda^3}{g} \right)^2 + \dots$$

Low- T , $z \rightarrow 1$ Degenerate Bose gas.

$f_{5/2}^+(z)$ increases w/ z in range $z \in [0, 1]$



Define $\zeta_m = f_m^+(1) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{e^x - 1} dx$

Integral $\sim x^{m-2}$ as $x \rightarrow 0$

$\rightarrow \zeta_m$ finite for $m > 1$
infinite for $m \leq 1$.

Can show $\frac{d}{dz} f_m^+(z) = \frac{1}{z} f_{m-1}^+(z)$

\rightarrow For any m , a sufficiently high derivative of $f_m^+(z)$ will diverge at $z=1$.

Density of excited states $\left[\begin{array}{l} \text{Ground state is separated in} \\ \langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1}, \text{ but } \vec{k} = \vec{0} \\ \text{state must be included separately in} \\ \text{the integral approximation} \end{array} \right.$

$$n_x = \frac{g}{\lambda^3} f_{3/2}^+(z)$$

$$n_x \leq n^* = \frac{g}{\lambda^3} \zeta_{3/2} \propto T^{3/2}$$

$$N = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

$$\approx \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1}$$

$$\text{At } \ln z \sim -1, \quad \frac{n \lambda^3}{g} = \frac{n}{g} \left(\frac{h}{\sqrt{2\pi m k_B T}} \right)^3 \ll \zeta_{3/2} = 2.612 \dots$$

$$n_x \approx n$$

Lowering T , the limiting density of excited states is reached when

$$\frac{n \lambda(T_c)^3}{g} = \zeta_{3/2}$$

$$\rightarrow T = T_c(n) = \frac{h^2}{2\pi m k_B} \left(\frac{n}{g \zeta_{3/2}} \right)$$

Bose-Einstein Condensation: $T < T_c$

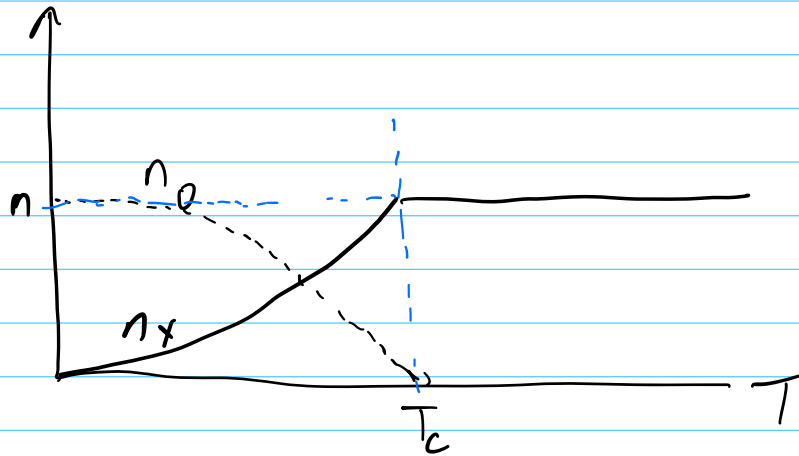
For all $T < T_c$, $z \approx 1$, i.e. $\mu \approx 0$.

$$\text{Density of excited states } n_x = n^* = \frac{g \zeta_{3/2}}{\lambda^3} \propto T^{3/2}$$

\hookrightarrow

$\rightarrow n_0 = n - n_x = \text{density of gas particles in } \vec{k} = \vec{0} \text{ ground state.}$

Macroscopic occupation of lowest-energy state
 = Bose-Einstein Condensation



$$T=0: n_x = n^* = 0 \quad \rightarrow \quad n_0 = n$$

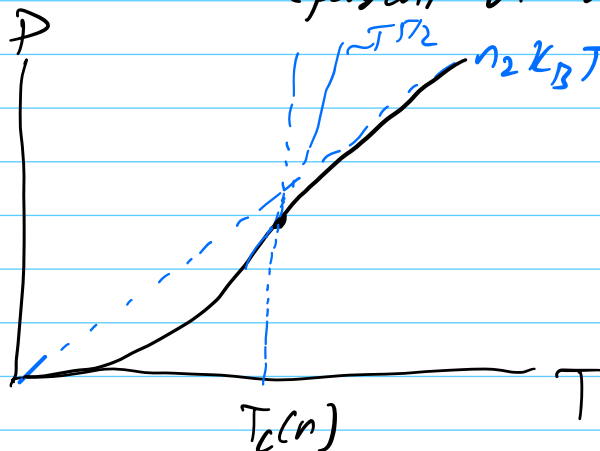
All gas particles in ground state.

Gas pressure for $T < T_c$:

$$\beta P_+ = \frac{g}{\lambda^3} f_{5/2}^+(1) = \frac{g}{\lambda^3} \zeta_{5/2} \approx 1.341 \frac{g}{\lambda^3}$$

$$P_+ \propto T^{5/2} \quad \text{as } T \rightarrow 0.$$

— independent of density n .



Compressibility:

$$\kappa_T = \frac{1}{n} \left. \frac{\partial n}{\partial P} \right|_T$$

$$\text{Use } \frac{dP}{dz} = \frac{g \kappa_B}{\lambda^3} \frac{1}{z} f_{3/2}^+(z)$$

$$\frac{dn}{dz} = \frac{g}{\lambda^3} \frac{1}{z} f_{1/2}^+(z)$$

$$\rightarrow \kappa_T = \frac{1}{n} \frac{dn/dz}{dP/dz}$$

$$= \frac{f_{1/2}^+(z)}{n \kappa_B T f_{3/2}^+(z)}$$

$$f_{1/2}^+(z) \rightarrow \infty \text{ as } z \rightarrow 1, \text{ so}$$

$$\kappa_T \rightarrow \infty \text{ as } T \rightarrow T_C \text{ from above.}$$

Heat Capacity

$$E = \frac{3}{2} PV = \frac{3}{2} V \frac{g}{\lambda^3} \kappa_B T f_{5/2}^+(z)$$

$$\propto T^{5/2} f_{5/2}^+(z)$$

$$C_V = \left. \frac{dE}{dT} \right|_{V,N} = \frac{3}{2} V \frac{g}{\lambda^3} \kappa_B T \left[\frac{5}{2T} f_{5/2}^+(z) + \frac{1}{z} f_{3/2}^+(z) \frac{dz}{dT} \right]$$

To find $\left. \frac{dz}{dT} \right|_{V, N}$, use

$$\left. \frac{dN}{dT} \right|_V = 0 = \frac{g}{\lambda^3} V \left[\frac{3}{2T} f_{3/2}^+(z) + \frac{1}{z} f_{1/2}^+(z) \frac{dz}{dT} \right]_{V, N}$$

$$\rightarrow \left. \frac{T}{z} \frac{dz}{dT} \right|_{V, N} = - \frac{3}{2} \frac{f_{3/2}^+(z)}{f_{1/2}^+(z)}$$

Then,
$$C_V = \frac{3}{2} V \frac{g}{\lambda^3} \left[\frac{5}{2} f_{5/2}^+(z) - \frac{3}{2} \frac{f_{3/2}^+(z)^2}{f_{1/2}^+(z)} \right]$$

High-T, low n : $z \ll 1$

$$\frac{C_V}{Nk_B} = \frac{3}{2} \left[1 + n \frac{\lambda^3}{2^{7/2}} + \dots \right] > \frac{3}{2}$$

↑
classical value.
 $T \rightarrow \infty$

low-T: $z=1$

$$\frac{C_V}{Nk_B} = \frac{3}{2} \frac{g}{n \lambda^3} \lambda^3 \left[\frac{5}{2} \zeta_{5/2}(1) - \frac{3}{2} \frac{\zeta_{3/2}(1)}{\zeta_{1/2}(1)} \right]$$

↑
because $\zeta_{1/2}(1) = \infty$.

$$= \frac{15}{4} \frac{g}{n \lambda^3} \zeta_{5/2}$$

$$= \frac{15}{4} \zeta_{5/2} \cdot \frac{1}{\zeta_{3/2}} \left(\frac{T}{T_c} \right)^{3/2} \sim n \frac{\lambda(T_c)^3}{g} = \zeta_{3/2}$$

Hence, $C_V \propto T^{3/2}$ for $T < T_C$

