Kardar 6.3

Blackbody Radiation

Normal modes of the electromagnetic field = electromagnetic waves.

Modes are labeled by wavevector $\vec{k}$, two polarizations $\alpha = 1, 2$.
$\nabla \cdot \vec{E} = 0 \rightarrow$ solutions to Maxwell's equations of the form
$\vec{E} = \vec{\varepsilon} \cos(\vec{k} \cdot \vec{x} - \omega t)$ have $\vec{\varepsilon} \cdot \vec{k} = 0$ — transverse

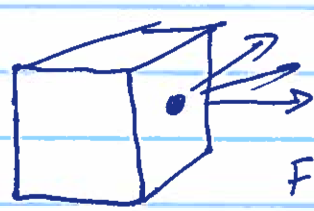
Each mode oscillates, and in appropriate coordinates is described by a harmonic oscillator Hamiltonian:

$$\mathcal{H} = \frac{1}{2} \sum_{k,\alpha} \left[|\tilde{p}_{\vec{k}\alpha}|^2 + \omega_\alpha(\vec{k})^2 |\tilde{u}_\alpha(\vec{k})|^2 \right],$$

with $\omega_\alpha(\vec{k}) = ck$, c = speed of light

Periodic boundary conditions $\rightarrow \vec{k} = 2\pi \left(n_x/L_x, n_y/L_y, n_z/L_z \right)$.
— similar to phonons in solid, but without a Brillouin zone.

Classically: $k_B T$ energy per mode, infinite number of modes because n_x, n_y, n_z can be arbitrarily large
$\rightarrow$ Infinite energy density from high-frequency modes!
— Ultraviolet catastrophe!



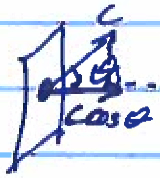
Allow light to escape from a small hole in box at temp. T.

$$\text{Flux} = \frac{\text{Escaping energy}}{(\text{Unit Area})(\text{Unit time})}, \quad \phi = \langle c_\perp \rangle \frac{E}{V}$$

component of velocity $\perp$ hole.

$$\langle c_\perp \rangle = c \cdot \frac{1}{4\pi}\int_0^{\pi/2} \underset{\uparrow}{2\pi} \sin\theta\, d\theta \cos\theta = \frac{c}{4}$$

from $\int_0^{2\pi} d\phi$ azimuthal angle

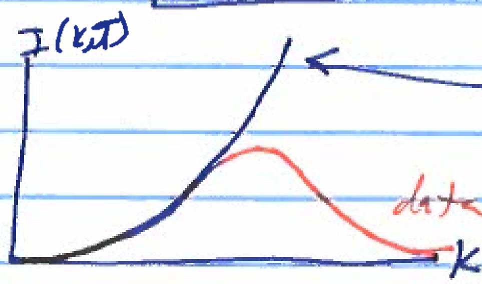


$$\phi = \frac{1}{4} c \frac{E}{V} = \frac{1}{4} \frac{c}{V}\cdot \sum_{k} (k_B T)\cdot \overset{\text{polarizations}}{2} = \frac{1}{4}\frac{c}{V}\cdot V\int \frac{d^3k}{(2\pi)^3}(k_B T)\cdot 2$$

$$= \frac{1}{4} c (k_B T)\cdot \frac{2\cdot 4\pi}{(2\pi)^3}\int dk\, k^2 = \frac{c}{4\pi^2} k_B T \int dk\, k^2$$

Write $\phi = \int dk\, I(k,T)$, where the intensity per unit wavevector is (spectral radiance)

$$\boxed{I^{\text{classical}}(k,T) = \frac{c}{4\pi^2} k_B T\, k^2}$$ Rayleigh-Jeans Law



To reproduce the measured blackbody spectrum, Max Planck suggested that the allowed energies in the EM modes are

$$\mathcal{H}^\gamma = \sum_{k,\alpha} \hbar c k\left(n_\alpha(\vec{k}) + \tfrac{1}{2}\right) \quad \text{with } n_\alpha(\vec{k}) = 0,1,2,\ldots$$

Now $$E = \langle \mathcal{H}^\gamma \rangle = \sum_{k,\alpha} \hbar c k \left(\frac{1}{2} + \underbrace{\frac{e^{-\beta\hbar c k}}{1-e^{-\beta \hbar c k}}}_{\langle n_\alpha(\vec{k})\rangle}\right)$$

$$= E_0 + \frac{2V}{(2\pi)^3}\int d^3k\, \frac{\hbar c k}{e^{\beta\hbar c k} - 1}$$

$$E = E_0 + V \cdot \frac{\hbar c}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^4 \int_0^\infty dx \, \frac{x^3}{e^x - 1} \qquad (x = \beta\hbar c k)$$

$$= E_0 + V \cdot \left(\frac{k_B T}{\hbar c}\right)^3 k_B T \, \frac{\pi^2}{15}$$

Dropping the (infinite) zero-point energy E_0, the flux of escaping energy from a container is

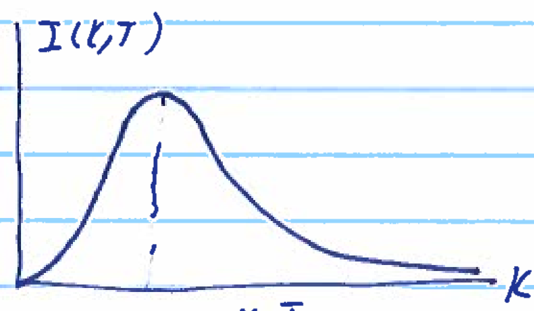
$$\phi = \langle c_\perp \rangle \frac{E}{V} = \frac{1}{4} c \frac{E}{V} = \underbrace{\frac{\pi^2}{60}\frac{k_B^4}{\hbar^3 c^2}}_{\text{Stefan-Boltzmann const. } \sigma = \frac{\pi^2}{60}\frac{k_B^4}{\hbar^3 c^2} \approx 5.67\times 10^{-8} \, W/m^2 K^4} T^4 \qquad \text{Finite!}$$

If we write $\phi = \int dk \, I(k,T)$, the blackbody spectrum is

$$\boxed{I(k,T) = \frac{\hbar c^2}{4\pi^2} \frac{k^3}{e^{\beta\hbar c k} - 1}}$$

$k \gg \frac{k_B T}{\hbar c}$: $\quad I(k,T) \to \hbar c^2 k^3 e^{-\beta\hbar c k} / 4\pi^2$

$k \ll \frac{k_B T}{\hbar c}$: $\quad I(k,T) \to \frac{c k_B T k^2}{4\pi^2}$ $\qquad$ classical result



$k^*(T) \sim \frac{k_B T}{\hbar c}$

NO ultraviolet catastrophe because energy difference between excitations at large k grows with k.

$\hbar\omega_k \cdot n_\alpha(\vec{k}) = \hbar c k \, n_\alpha(\vec{k})$

Pressure in EM Radiation:

$$Z = \sum_{\{n_\alpha(\vec{k})\}} \prod_{\vec{k},\alpha} \exp\left[-\beta\hbar\omega(\vec{k})\left(n_\alpha(\vec{k})+\tfrac{1}{2}\right)\right] = \prod_{\vec{k},\alpha} \frac{e^{-\beta\hbar ck/2}}{1-e^{-\beta\hbar ck}}$$

Free energy $F = -k_B T \ln Z = k_B T \sum_{\vec{k},\alpha}\left[\frac{\beta\hbar ck}{2} + \ln\left(1-e^{-\beta\hbar ck}\right)\right]$

$$= \underset{\uparrow \atop \alpha=1,2}{2}V\int\frac{d^3k}{(2\pi)^3}\left[\frac{\hbar ck}{2} + k_B T\ln\left(1-e^{-\beta\hbar ck}\right)\right]$$

Pressure $P = -\frac{\partial F}{\partial V}\Big|_T = -\int\frac{d^3k}{(2\pi)^3}\left[\hbar ck + 2k_B T\ln\left(1-e^{-\beta\hbar ck}\right)\right]$

zero-point pressure (divergent)

$$= P_0 - \frac{k_B T}{\pi^2}\int_0^\infty dk\, k^2\ln\left(1-e^{-\beta\hbar ck}\right)$$

$$\overset{\text{by parts}}{=} P_0 + \frac{k_B T}{\pi^2}\int_0^\infty dk\,\frac{k^3}{3}\cdot\frac{\beta\hbar c\, e^{-\beta\hbar ck}}{1-e^{-\beta\hbar ck}}$$

Integrand has same form as in calculation of internal energy E:

$$\boxed{P = P_0 + \frac{1}{3}\frac{E}{V}}$$

$P_0 = -\int\frac{d^3k}{(2\pi)^3}\hbar ck = -\frac{E_0}{V}$ vacuum pressure, independent of T.

- The vacuum is like a negative-pressure fluid
  $\rightarrow$ cosmological constant, Casimir effect

Extra pressure is of the form $P = \frac{1}{3}E/V$
- relativistic gas with $\varepsilon \propto |\vec{p}|$ in 3 Dimensions.