The Laplace-Runge-Lenz Vector

In addition to the angular momentum, the Kepler problem (motion in a $\frac{1}{r^2}$ attractive force law) features another conserved vector.

Suppose the law of motion can be written

$$\dot{\vec{p}} = f(r)\,\frac{\vec{r}}{r}.$$

$\hat{r} = \frac{\vec{r}}{r}$

Then

$$\dot{\vec{p}} \times \vec{L} = \frac{f(r)}{r}\,\vec{r} \times (m\vec{r} \times \dot{\vec{r}})$$

$$= m\frac{f(r)}{r}\left[\vec{r}(\vec{r}\cdot\dot{\vec{r}}) - r^2\dot{\vec{r}}\right]$$

$$= m\frac{f(r)}{r}\left[\vec{r}\,\frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) - r^2\dot{\vec{r}}\right]$$

$$= m\frac{f(r)}{r}\left[\vec{r}\,\frac{2}{2}r\dot{r} - r^2\dot{\vec{r}}\right]$$

$$= -m f(r) r^2\left[\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\,\dot{r}}{r^2}\right]$$

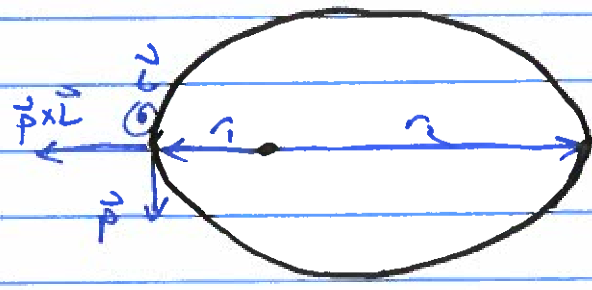
$$= -m f(r) r^2 \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

For $f(r) = -\frac{K}{r^2}$, with $\frac{d\vec{L}}{dt} = 0$:

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = +mK\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\rightarrow \frac{d\vec{A}}{dt} = 0, \text{ where } \vec{A} \equiv \vec{p}\times\vec{L} - mK\frac{\vec{r}}{r}$$

Laplace-Runge-Lenz Vector.



$\leftarrow \vec{A} = \vec{p}\times\vec{L} - mK\frac{\vec{r}}{r}$

$\vec{A}$ points along the radial direction to the perihelion.

Note that $\vec{A}\cdot\vec{L} = \vec{0}$.

perihelion at $\theta = 0$.

$$\vec{A}\cdot\vec{r} = Ar\cos\theta = \vec{r}\cdot\left(\vec{p}\times\vec{L} - mK\frac{\vec{r}}{r}\right)$$

$$= \vec{L}\cdot(\vec{r}\times\vec{p}) - mKr$$

$$= \ell^2 - mKr$$

$$\Rightarrow \boxed{\frac{1}{r} = \frac{mK}{\ell^2}\left(1 + \frac{A}{mK}\cos\theta\right)} \quad \text{ellipse}$$

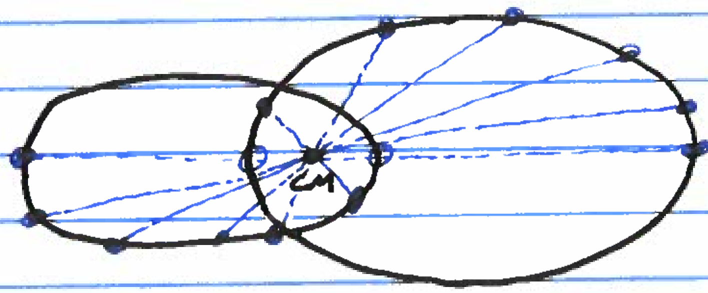
$\uparrow$ eccentricity $e = \frac{A}{mK}$

We previously had $e = \sqrt{1 + \frac{2E\ell^2}{mK^2}}$

$\Rightarrow$ $\vec{A}$ is not independent of the conserved quantities E and $\vec{L}$.

Note that we have been describing the motion about the center of mass in terms of the radial distance r between the two masses

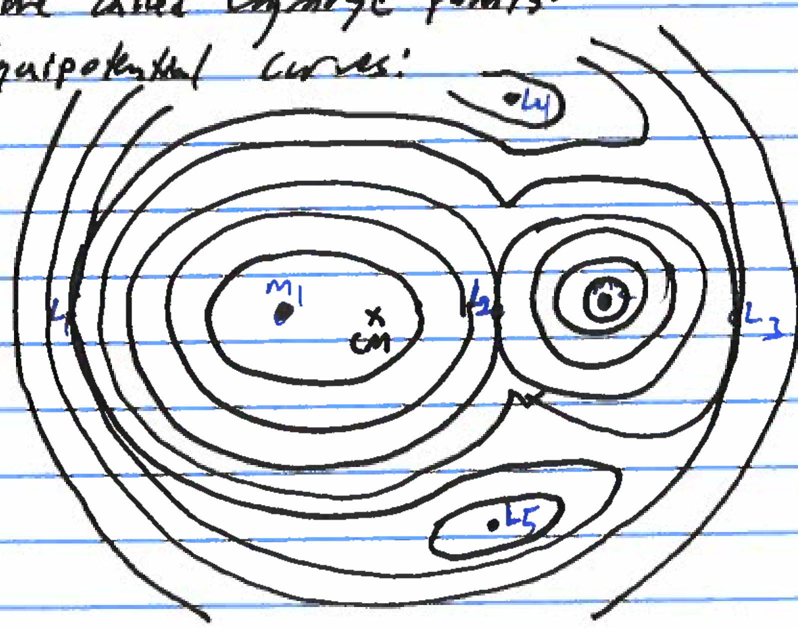
Each of the two masses orbits the center of mass.



Lagrange points

Suppose we are interested in the motion of a small satellite in the background of two massive objects, e.g. the Earth-moon or the Earth-sun system.
There are two stable and three unstable satellite positions that remain fixed relative to the two masses.
They are called Lagrange points.

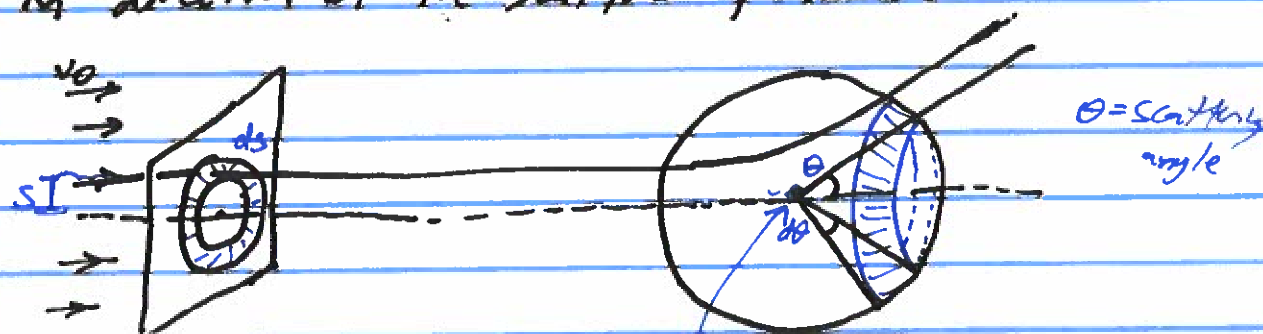
Equipotential curves:



Scattering in a Central Force Field - Goldstein 3.10

1-body scattering: Scattering off of a center of force

Consider a uniform beam of particles incident on the center of force.
We assume that the force decreases toward zero as the particles approach infinity.
We characterize the scattering in terms of the change in direction of the scattered particle.



Intensity of beam = flux density = # particles/unit time/unit area

Cross section for scattering in a given direction
$\equiv$ Differential scattering cross section
$= \sigma(\Omega)$, Ω = solid angle of scattering

$$\sigma(\Omega)\,d\Omega = \frac{\#\text{ particles scattered into solid }\measuredangle\ d\Omega\ /\text{unit time}}{\text{incident intensity.}}$$

Note that the dimension of $\sigma(\Omega)$ is area.

With central forces there is symmetry of scattering around the axis of the incident beam.
Scattering is then characterized by the scattering angle θ.

$$d\Omega = 2\pi \sin\theta \, d\theta$$

In terms of the impact parameter s, i.e. the perpendicular distance between the center of force and the incident velocity, the angular momentum (conserved) is

initial particle speed.

$$\ell = m v_0 s = s\sqrt{2mE}$$

energy of particle (conserved)

The scattering angle θ depends on s and E.

Suppose each s corresponds to a unique θ (otherwise we will need appropriate sums in what follows):
Suppose particles between s and $s+ds$ scatter into angles between θ and $\theta + d\theta$.

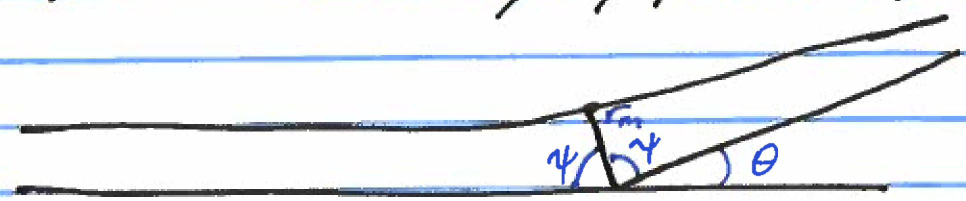
particles/unit time through annulus of size ds
= # particles/unit time passing through corresponding solid $\angle$.

$$2\pi s |ds| \, I = \sigma(\theta) \cdot 2\pi \sin\theta |d\theta| \cdot I$$

$$\Rightarrow \boxed{\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|} \text{, where } s = s(\theta, E).$$

The orbit is symmetric about the direction of the periapsis (closest approach to the scattering center).

Define $\psi = \measuredangle$ between incoming asymptote and periapsis



$\Theta = \pi - 2\psi$

r_m = distance of closest approach

We can borrow from our earlier analysis of the Kepler problem, in which we found the angle traversed as a function of r:

$$\psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r)}{\ell^2} - \frac{1}{r^2}}}$$

Using $\ell = s\sqrt{2mE}$,

$$\psi = \int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1-\frac{V(r)}{E}\right) - s^2}}$$

$u = 1/r$

$$= \int_0^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(u^{-1})}{E} - s^2u^2}}$$

The scattering ∡ in terms of s and E is then:

$$\boxed{\Theta(s) = \pi - 2\int_0^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(u^{-1})}{E} - s^2u^2}}}$$

With $\Theta(s)$ in hand we could then calculate the cross section $\sigma(\Theta)$.

Example: Repulsive scattering of charged particles by Coulomb field.

Two charged particles, $q_1 = -Ze$, $q_2 = -Z'e$
fixed charge ↑ ↖ scattered charge.

$$f = \frac{ZZ'e^2}{r^2} \equiv -\frac{K}{r^2}, \quad K = -ZZ'e^2$$

$E > 0 \rightarrow$ hyperbolic trajectory w/ eccentricity

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2Es}{ZZ'e^2}\right)^2}$$

using $l = s\sqrt{2mE}$

ϵ instead of e (electron charge)

with periapsis at $\phi = 0$, the orbit equation is

$$\boxed{\frac{1}{r} = \frac{mZZ'e^2}{l^2}(\epsilon\cos\phi - 1)}$$

Direction of incoming asymptote $r \rightarrow \infty$, $\phi \rightarrow \Psi$

$$\boxed{\cos\Psi = 1/\epsilon}$$

$\Theta = \pi - 2\Psi$ scattering angle

$\cos\Psi = 1/\epsilon \rightarrow \cos\left(\frac{\pi-\theta}{2}\right) = \boxed{\sin\frac{\theta}{2} = 1/\epsilon}$

This gives $\theta(s)$:

$$\cot^2\frac{\theta}{2} = \epsilon^2 - 1 = \left(1 + \left(\frac{2Es}{ZZ'e^2}\right)^2\right) - 1$$

$$= \left(\frac{2Es}{ZZ'e^2}\right)^2$$

$$\rightarrow \boxed{s = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}}$$

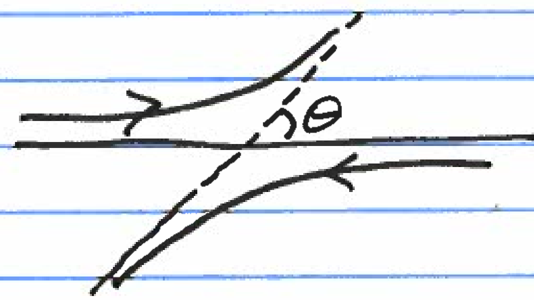
Then $\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$

$$= \left(\frac{ZZ'e^2}{2E}\cot\theta/2\right) \cdot \frac{1}{2\sin\theta/2\cos\theta/2} \cdot \frac{ZZ'e^2}{2E}\frac{1}{\sin^2\theta/2}\cdot\frac{1}{2}$$

$$\boxed{\sigma(\theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2\csc^4\frac{\theta}{2}}$$ Rutherford Scattering Cross section

If we consider scattering of two particles off of one another without considering one particle fixed (which is only a good approximation if the particle is much more massive than the scattered particle), then $m \to \mu = \frac{m_1 m_2}{m_1 + m_2}$ and Θ is the scattering angle

in the center of mass frame.



Note that we are often interested in a laboratory frame in which one of the particles is initially at rest.

In that case, we would need to consider the angle of scattering of the initially moving particle, which is not the same as Θ unless the initially stationary particles is infinitely massive.

We will not pursue this here.